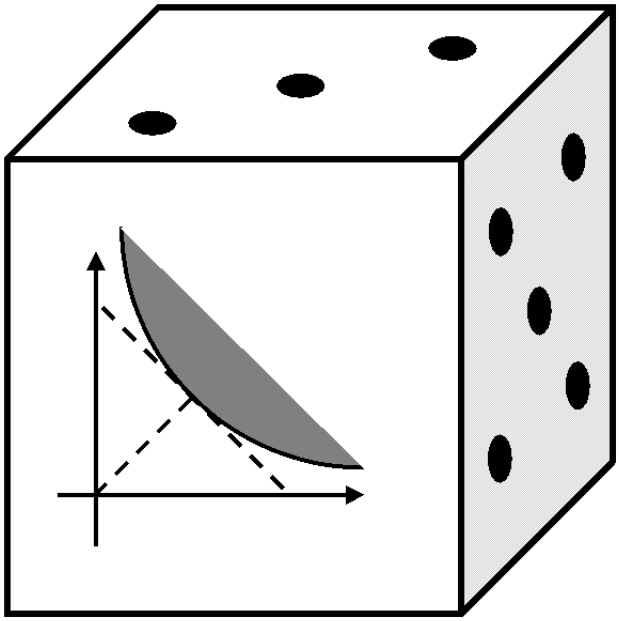
SESAM

USER MANUAL

Proban Theory



General Purpose Probabilistic Analysis Program

DET NORSKE VERITAS

SESAM
User Manual

Proban Theory

General Purpose Probabilistic Analysis Program

October 1st, 2004

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1 INTRODUCTION

Proban provides the industry with an advanced probabilistic analysis tool, i.e. a program for the computation of probabilities, crossing rates, distributions and sensitivity factors associated with structural safety. Proban also includes methods which covers demands in applications such as economic risk analysis, traditional risk and reliability analysis, stochastic reservoir simulation, operations analysis etc.

Hence, Proban includes the First Order Reliability Method, FORM, and the Second Order reliability Method, SORM, that has been introduced to solve the high reliability problems often encountered in studies of structural safety. These methods are theoretically justified by asymptotic theory. In many applications the probability is typically 10^{-3} or less, making the use of asymptotic methods attractive.

Through Proban a number of sampling methods are also made available. Those are the (adjusted) Monte Carlo simulation method, the Directional simulation method, the Axis-orthogonal simulation method, the Designpoint simulation method, and the Latin-Hypercube simulation method. The Monte Carlo simulation method and the Latin-Hypercube simulation method are efficient for sampling the central part of a distribution. The Designpoint simulation method, is also efficient for sampling probabilities in the tail of a distribution, provided the number of variables in the problem is not too large. The Axis-orthogonal simulation method samples a correction to the FORM-reliability for an intersection. The Designpoint simulation method samples the probability around the design point using an adjusted Monte Carlo simulation method.

The events that Proban can handle are method dependent. If the method relies on FORM or SORM, then the event can at most be a union of intersections of single events. If the method is the Monte Carlo simulation method or the Directional simulation method then an arbitrary composition of intersections and unions can be treated.

A variable can be modelled as a numeric constant, a function, a distribution, a time dependent stochastic process and the probability of an event. A parameter of a variable can be assigned a coordinate of an other variable so that a network structure for dependencies between variables can be defined. Additional statistical dependence between variables can be modelled through correlations.

A library of functions and a library of distributions are made available with Proban. Simple functions and integrals of functions can be input interactively. Thus quite complex statistical models can be modelled through the input.

The results of a probabilistic analysis are a) the probability of a stochastic process being in or crossing into an event domain, b) the crossing rate, c) the probability of an event conditioned on a measured variable, d) the partial derivative of the probability with respect to a measured value or a threshold value of the event function, together with sensitivity factors and uncertainty importance factors.

Alternatively the distribution of a variable or an event function as well as its first four moments, the mean, the standard deviation, the skewness and the kurtosis are estimated. The distribution may be restricted to an event domain. Sensitivity factors are provided for the statistical moments.

The purpose of this manual is to provide the user a detailed description of the methods implemented in Proban. Hopefully the manual will guide the user to an appropriate model for the problem at hand.

1.1 Proban in the SESAM System

SESAM is comprised of preprocessors, environmental analysis programs, structural analysis programs and post processors. An overview of SESAM is shown in Figure 1.1.

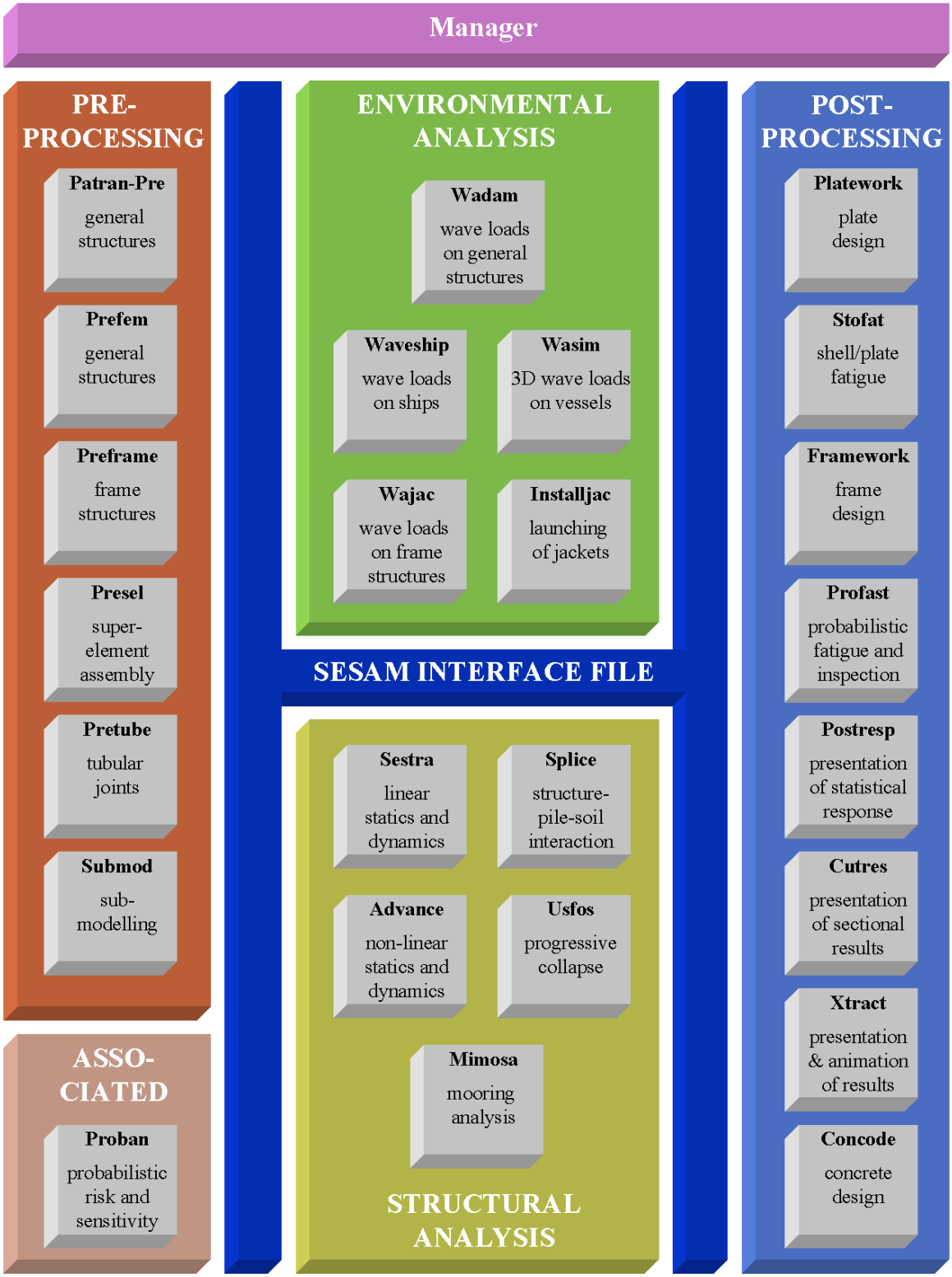


Figure 1.1 SESAM overview

1.2 How to read the Manual

- Chapter 2 describes the models for variables that Proban can handle.
- Chapter 3 describes the models for events that Proban can handle.
- Chapter 4 describes the results that Proban delivers.
- Chapter 5 is a guide to the methods.
- Chapter 6 describes the mapping of the random model variables to a set of independent standard normal random variables.
- Chapter 7 to Chapter 22 describes the methods implemented in Proban.
- Appendix A to Appendix E gives more specialized implementation notes.
- Appendix F tabulates the terminology used in this manual.
- Appendix REFERENCES lists the literature referenced in this manual.

2 PROBABILISTIC MODEL - VARIABLES

2.1 Definition of Variables

A stochastic model is defined through variables. A variable describes a functional relationship in the physical model or the randomness of a parameter in the model.

A variable has the following attributes, a *name*, a *description*, a *type* and a *parameter list*. The name of a variable is a user-defined attribute which is unique among the variables and is used to reference the variable. The description of a variable is a user-defined attribute which is output when the variable is printed. The type of variable identifies how its value is calculated. The parameter list is a set of user-defined values which are input to the specific calculation method.

A variable **X** is a vector (X_1, X_2,..., X_n) of *dimension n* and *coordinates* X_i, i=1,2,...,n

A number of assignments can be made to change the calculation method for variables. Thus assignment of extreme values (max/min of a number of independent realizations) and assignment of correlation coefficients are available.

Assignments can also change the interpretation of a variable. Thus a variable can be assigned as a time derivative process or as a measured value. Its integration level in an analysis of a time dependent stochastic process can be changed. It can be specified as a parameter for a generated distribution variable and for an event probability variable.

A parameter of a variable is assigned a numeric constant or a coordinate of another variable. This is permissible for as long as a variable does not (in)directly reference itself. Thus, Proban handles a network structure of dependencies between variables.

A model can thus split into several variables that are modelled and studied separately and also assembled to study the joint behaviour.

Example: The variable X has the Lognormal distribution with mean $\mu = y$ (value of variable *Y*), standard deviation $\sigma = z_3$ (value of coordinate 3 of variable *Z*) and lower bound $\gamma = 0.5$. The type attribute of *X* is the Lognormal distribution and the parameter list to be used when results are required for *X* is μ, σ and γ.

Variables are created by using the command CREATE VARIABLE.

Variables are changed by using the command CHANGE VARIABLE.

Variables are deleted by using the command DELETE VARIABLE.

2.2 Types of Variables

2.2.1 General

A variable has one of the following types: *Fixed* value, *distribution*, *fitted distribution*, event *probability*, *function*, *generated* distribution, and *time*.

2.2.2 Distribution

The variable is assigned a distribution from the distribution library, e.g., the exponential distribution, the normal distribution etc. A distribution describes the randomness of a parameter in the model. The distributions available in Proban are described in the Proban Distribution Manual.

A distribution variable may be assigned to parameters of any other variable.

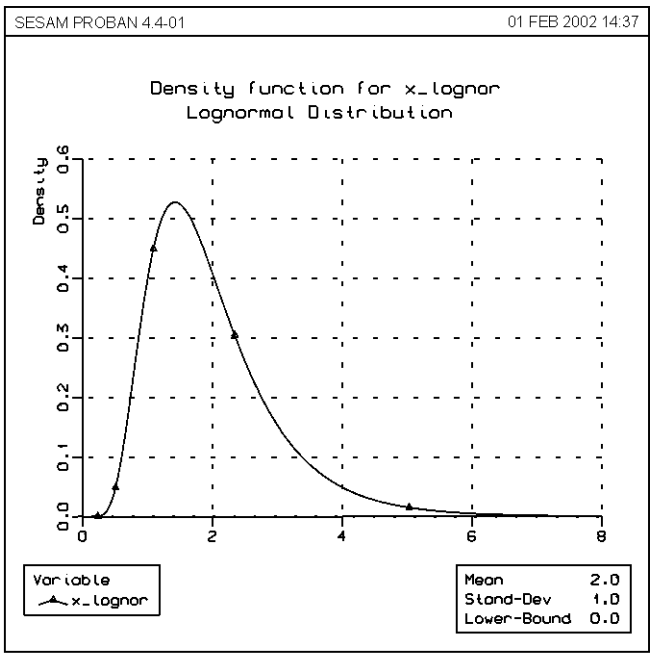


Figure 2.1 Lognormal Distribution

2.2.3 Event Probability

A probability variable is assigned the probability of an event *E* (see Chapter 3). The probability is delivered in terms of the *probability*, the log_{10} of the probability or the *reliability index*, see Section 4.3.

The event probability may be conditioned on the value of some of the variables in the event model. Those variables then work as parameters for the probability variable.

If the variables conditioned on are represented by the vector **Y**, then the event probability becomes $P_E(\mathbf{W};\mathbf{y})$ where **y** is a realisation of **Y** and **W** is the vector of variables defining *E* minus the variables **Y** conditioned on.

Fixed valued variables are always conditioned on and are not required to be specified as conditioning variables.

The variables **Y** cannot be dependent on the variables **W**. Thus in(direct) reference to **W** for the variables **Y** is not permitted. Also, correlation of variables in **Y** and variables in **W** is not permitted.

A probability variable invokes a two level *nested analysis* with the variables **Y** being random at the outer level and the variables **W** being random at the inner level. The analysis method is restricted to FORM. This restricts the models for the event *E* to be models that can be treated by FORM, see Chapter 7. The outer level analysis permits one single event only. At most one probability variable can be active in a model calculated on.

The variables **W** must permit a FORM analysis to take place. For example, the result of the conditioning assignment must not make a subevent *E* non-random or without a FORM solution.

The conditioning variables **Y** are assigned by using the command ASSIGN CONDITIONING.

2.2.4 Fitted Distribution

The variable is created by fitting the parameters of a library distribution to data. The data may be input by user or be generated by Proban in an analysis. The variable is then employed in further analyses just like a distribution variable.

A fitted distribution variable may be assigned to parameters of any other variable.

The fitting methods are:

- The (*weighted*) *least squares fit* which finds the distribution parameters α which minimises the function $\Sigma w_i(F(x_i;\alpha) - P_i)^2$. Here w_i is the weight of the argument x_i, P_i is the corresponding probability and $F()$.is the fitted distribution function).
- The (*weighted*) *maximum likelihood fit* which finds the number of distribution parameters α which minimises the function $\Sigma w_i f(x_i;\alpha)$. Here w_i is the weight of the observation x_i and $f()$ is the fitted density distribution.
- The (*weighted*) *moment fit* which calculates the first few moments of the (weighted) observations and calculates the distribution parameters from the moment formulas.

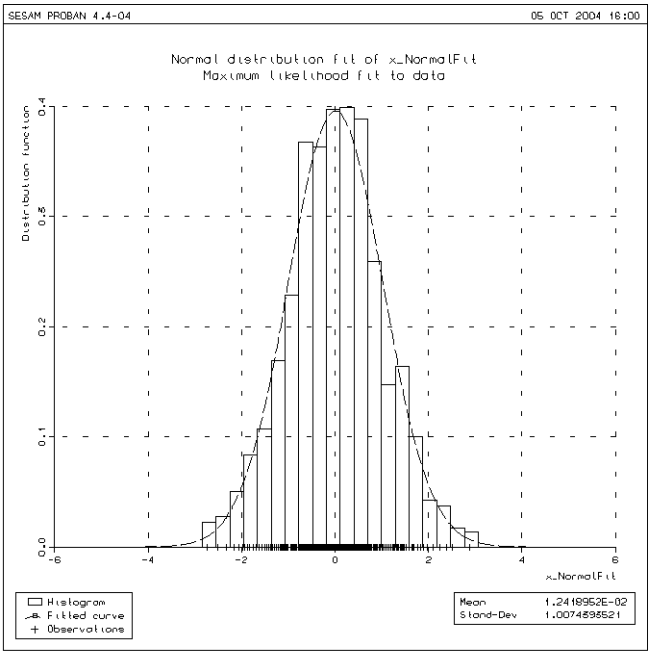


Figure 2.2 Fitted Distribution

2.2.5 Fixed Value

The variable is a constant value, for example 0.5.

A fixed variable may be assigned to parameters of every other variable. It permits sensitivity factor calculations and parameter studies for parameters which enters the physical model at more than one place.

2.2.6 Function

A function variable is assigned a function from the functions library.

Proban has a built-in library of simple mathematical functions. Simple one line formula functions as well as one-dimensional integrals of a function can be created on input. Response fits to user defined or library functions can also be created on input. Created functions can reference other functions, also functions created on input. Thus quite complex functions can be modelled. More elaborate functions can be coded and linked into the program in a user defined library of functions.

A variable which has the function type attribute may be assigned to parameters of any other variable.

2.2.7 Generated Distribution

A generated distribution variable Z is assigned the distribution of another variable X. This means that randomness and functional relationships of the variables in the model for X defines the functional form of Z.

The variable Z may be conditioned on the values of some of the variables in the model for X, say **Y**. The distribution then becomes X(**W**;**y**) where **y** is a realization of **Y** and **W** is the vector of variables in the model for X excluding the variables **Y**.

Fixed valued variables are always conditioned on and are not required to be specified as conditioning variables.

The distribution of $Z(\mathbf{y})$ has the functional form of the distribution $X(\mathbf{W};\mathbf{y})$ and the vector **y** is the parameters for the distribution. Two variables $Z_1(\mathbf{y}_1)$ and $Z_2(\mathbf{y}_2)$ are thus independent unless they are explicitly correlated through specification of a correlation coefficient or through dependency between $\mathbf{Y}_1$ and $\mathbf{Y}_2$.

Variables of type generated distribution may be mutually correlated and also correlated with variables of type distribution. A generated distribution variable may be assigned a (max/min of n) extreme value

The generated distribution invokes a FORM analysis for calculation of cumulative probabilities and for distribution fractiles. The model for $X(\mathbf{W};\mathbf{y})$ must therefore comply with the requirements for FORM.

If a variable of **W** is assigned a time derivative process, then the time derivative process is neglected within the generated distribution.

The conditioning variables **Y** are assigned by using the command ASSIGN CONDITIONING.

Variables of type probability and type generated distribution are not permitted in **W**.

2.2.8 Process Time

A time variable has two parameters, a) the *starting time* of a stochastic process and b) its *duration*.

If a time variable is assigned as an argument of another variable, then the value of the time parameter is installed, for example during integration over time.

The time variable is required only if there is explicit time dependency in the model. If explicit time dependency is not present, then the process is assumed stationary and the duration, if required, is defined outside the time variable.

Only one time variable can be defined for a model.

A probability variable cannot be used with models that involves a time dependent stochastic process.

2.3 Correlation of Random Variables

In many cases of probabilistic analysis, the available information about the variables in the model are the marginal distributions and their mutual correlations. The joint distributions are unknown.

In general there is an infinite number of joint distributions that leads to the same correlation of the variables. Therefore, it may not be possible to point out one distribution and state that this is the correct one. The probabilistic model information is generally *incomplete*.

To be able to treat distribution models with incomplete information, one has to decide the type of joint distribution. Der Kiureghian and Liu (1986) suggest the Nataf model, Nataf (1962), and shows for a variety of

cases that the model yields good results. The basic idea of the Nataf correlation model is that if each random variable is mapped onto a standard normal random variable, the joint distribution becomes the Multivariate Normal distribution.

The Nataf model may also be a complete distribution model.

Univariate variables of type *distribution* and type *generated* may be correlated.

If each variable is a random normal variable, then the joint distribution becomes the Multinormal distribution which is also available in the Proban distributions library. The joint distribution is then

$$f(\mathbf{x}) = (2\pi)^{-n/2} det(\mathbf{R})^{-1/2} \exp\left(-\frac{1}{2}\mathbf{v}^T\mathbf{R}^{-1}\mathbf{v}\right)\frac{\mathrm{d}v_1}{\mathrm{d}x_1}\cdots\frac{\mathrm{d}v_n}{\mathrm{d}x_n} \tag{2.1}$$

in which **R** is the correlation matrix for the standard normal variables **V**,

$$v_i = \Phi^{-1}(F_{X_i}(x_i)) \tag{2.2}$$

and

$$\frac{\mathrm{d}v_i}{\mathrm{d}x_i} = \frac{f_{X_i}(x_i)}{\varphi(v_i)} \tag{2.3}$$

Here $f_{X_i}(x_i)$ and $F_{X_i}(x_i)$ are the density function for X_i and the cumulative distribution function for X_i respectively. The functions $\phi(x)$ and $\Phi(x)$ are the standard normal density distribution function and the standard normal cumulative distribution function with x as argument, respectively.

If the variables are both of type distribution, then the correlation of the variables may be given directly (option BASIC). In all other cases, the correlation coefficient must be given for the corresponding standard normal random variables (option NORMALIZED).

A method for computing $\rho_{\mathbf{V}}$ from the correlation $\rho_{\mathbf{X}}$ of X_1 and X_2 and vice versa has been developed. Winterstein et al. (1989) use an expansion in Hermite polynomials. This permits the generality of the correlation model.

A coefficient of correlation may be a coordinate of a variable if the variable does not (in)directly reference a time dependent stochastic process.

If a variable is assigned an extreme value (max/min of a number of independent realizations) then the correlation is between the extreme value distributions.

The Nataf correlation model is input using the command ASSIGN CORRELATION.

2.4 Extreme (Max/Min) of a Number of Random variables

If a variable X is univariate and of type distribution, fitted distribution or generated distribution, then it can be redefined to represent the distribution of the maximum or minimum of n independent realizations of X. If X is distributed $F_X(x)$ then the distribution of X is

$$Y \sim F_X(y)^n \tag{2.4}$$

while the distribution of Z representing the minimum of n independent realizations of X is

$$Z \sim 1 - (1 - F_X(z))^n \tag{2.5}$$

The number n is a positive integer.

The max- and min- extreme value distributions are defined by using the command ASSIGN EXTREME.

2.5 Time Derivative Process

In order to define a time dependent stochastic process $X(t)$, $t \in [T_S, T_S + D]$ given by the variable X, a variable Y must be assigned as the time derivative process $\dot{X}(t)$, $t \in [T_S, T_S + D]$ of X.

The starting time T_S and the duration D are modelled as attributes assigned to the time variable if present.

A time derivative process must have zero mean. This is required for the methods to give correct results.

A model in which no explicit time dependency is defined, yields a stationary stochastic process with a time independent event boundary.

All other models yields either a non-stationary stochastic process or a time dependent event boundary.

Since any distribution from the distributions library may be assigned, the process $X(t)$ needs not be Gaussian.

Neither X nor Y can be variables of type probability, time or fixed.

The time derivative process has no meaning within the definition of a generated distribution.

A time derivative process Y is assigned to the process X by use of the command ASSIGN TIME-DERIVATIVE.

Proban includes a time derivative process in the analysis only if one runs a First passage Probability analysis or a Crossing Rate analysis.

2.6 Measured Value

A variable which represents the measured value of a stochastic process must be identified as the measured value for the equality event (see Chapter 3) which models that the underlying stochastic process has reached a measured value. For example that a crack has grown to the value a_m.

Restrictions: A measured value cannot be a variable of type time or type probability.

A measured value is assigned by use of the command ASSIGN MEASURED-VALUE.

3 PROBABILISTIC MODEL - EVENTS

3.1 Definition of Event

An event, $E(\mathbf{X})$, is a subset of the sample space for the stochastic process involved, i.e., a subset over all the possible outcomes of the stochastic process.

Calculation of a probability, a crossing rate, a derivative of a probability and a conditional distribution requires definition of an event.

An event is in Proban defined through a functional relationship

$$E(\mathbf{X}) = \{\mathbf{x}; G(\mathbf{x}) \leq 0\} \tag{3.1}$$

The event identifies the outcomes of interest while the random variables **X** define the nature of the stochastic process.

An event has the following attributes, a *name*, a *description*, a *type*, and a type dependent argument list. The name of an event is a user defined attribute which is unique among the events and is used to reference the event. The description of an event is a user defined attribute which is output when the event is printed.

Four event types are implemented in Proban, the *single event*, the *intersection* of events, the *union* of events and the *conditional* event.

The intersection of events and the union of events are collections of single events, unions of events and intersection of events. Thus an arbitrary network structure of events can be modelled. However, the models are restricted by the calculation method to be used. If the method is the Monte-Carlo simulation method or the Directional simulation method, then there is no restriction. If the method includes the FORM, then the model is restricted to a union of intersections of single events.

Events are created by use of the command CREATE EVENT.

Events are changed by use of the command CHANGE EVENT.

Events are deleted by use of the command DELETE EVENT.

3.2 Single event

The single event defines a functional relationship which identifies a subdomain of the state space. The single event $E_{SE}(\mathbf{X})$ is defined by

$$E_{SE}(\mathbf{X}) = \{\mathbf{x}; G(\mathbf{x}) \leq \theta\} \tag{3.2}$$

where $G(\mathbf{x})$ is a function and θ is a threshold value. The event is an *inequality event* which defines a volume in the *n*-dimensional *x*-space, the *event domain*.

Alternatively, the event is defined as

$$E_{SE}(\mathbf{X}) = \{\mathbf{x}; G(\mathbf{x}) = \theta\} \tag{3.3}$$

The event is an *equality event* which defines a volume in the $n-1$-dimensional *x*-space, the *event boundary*. To be meaningful the definition of an equality event requires the definition of a limit process. If a measurement variable is assigned to the event, the limit process is on the measured value. If not, the limit process is on $-\theta$, the threshold value.

The event function is

$$G_{SE}(\mathbf{X}) = G(\mathbf{x}) - \theta \tag{3.4}$$

In structural reliability the function $G_{SE}(\mathbf{x},\theta)$ is denoted as a *limit state function*, the domain of Equation (3.2), excluding the boundary is denoted as the *failure domain* or the *failure set*, the boundary is denoted as the *failure boundary* and the domain of $E_{SE}^{C}(\mathbf{X})$, the complement of $E_{SE}(\mathbf{X})$ is denoted as the *safe domain* or the *safe set*.

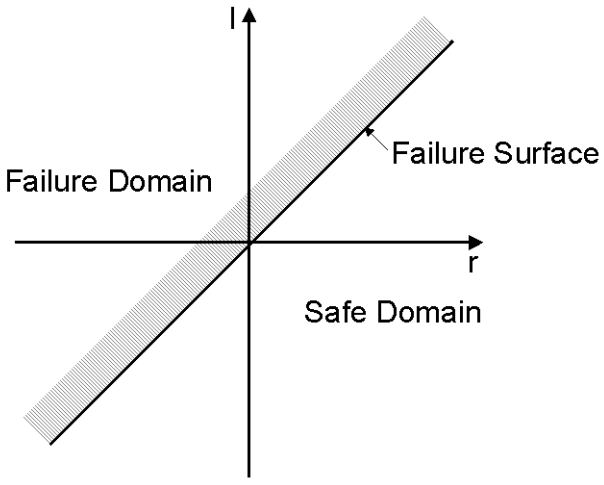


Figure 3.1 Load-Resistance Limit State function

A simple single event function is the load-resistance function $G(r,l)$ shown in Figure 3.1,

$$G(r,l) = r - l \tag{3.5}$$

The event of interest is failure. Thus the event is

$$E_{SE}(R,S) = \{(r,l); r - l \leq 0\} \tag{3.6}$$

3.3 Union of Events

The *union* of m events $E_i(\mathbf{X})$ is the collection of outcomes which are in at least one subevent $E_i(\mathbf{X})$:

$$E_{USE}(\mathbf{X}) = \bigcup_{i=1}^{m} E_i(\mathbf{X}) \tag{3.7}$$

The subevent $E_i(\mathbf{X})$ may be a single event, a union of events or an intersection of events.

A special case is the union of single events

$$E_{USE}(\mathbf{X}) = \bigcup_{i=1}^{m} E_{SE,i}(\mathbf{X}) \tag{3.8}$$

The importance of this event is that its probability can be calculated by use of bounding techniques or through a formulation in terms of complementary events.

Another special case is the union of intersections of single events.

$$E_{IUSE}(\mathbf{X}) = \bigcup_{i=1}^{m} E_{ISE,i}(\mathbf{X}) \tag{3.9}$$

The importance of this event is with the FORM and the SORM, since it is the most general system that can be treated with FORM and with SORM.

In structural reliability the event $E_{USE}(\mathbf{X})$ is used to model *series systems,* Figure 3.2.



Figure 3.2 Union of Events

3.4 Intersection of Events

The *intersection* of *m* events $E_i(\mathbf{X})$ is the collection of outcomes which are common to all the subevents $E_i(\mathbf{X})$:

$$E_I(\mathbf{X}) = \bigcap_{i=1}^{m} E_i(\mathbf{X}) \tag{3.10}$$

The subevent $E_i(\mathbf{X})$ may be a single event, a union of events or an intersection of events.

A special case is the intersection of single events

$$E_{ISE}(\mathbf{X}) = \bigcap_{i=1}^{m} E_{ISE,i}(\mathbf{X}) \tag{3.11}$$

The importance of this event is that it is the only intersection system for which the probability can be calculated by use of FORM or SORM.

In structural reliability the event $E_{ISE}(\mathbf{X})$ is used to model *parallel systems,* Figure 3.3.

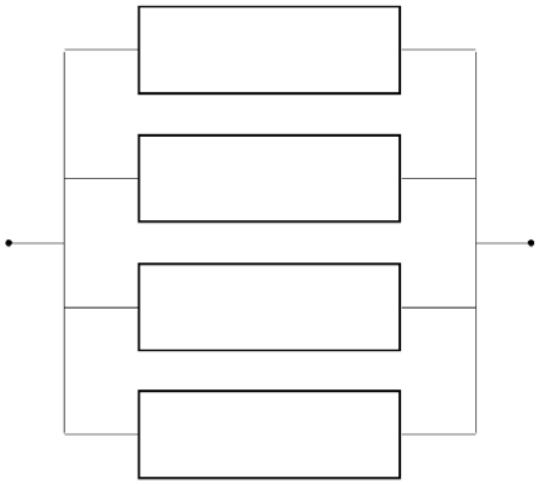


Figure 3.3 Intersection of Events

3.5 Conditional Event

The conditional event $E_C(\mathbf{X})$ is the collection of outcomes in $E_1(\mathbf{X})$ conditioned on the occurrence of the event $E_2(\mathbf{X})$:

$$E_C(\mathbf{X}) = (E_1 | E_2)(\mathbf{X}) \tag{3.12}$$

The subevent $E_1(\mathbf{X})$ or $E_2(\mathbf{X})$ is either a single event, a union or single events or an intersection of single events. The complexity of a subevent depends on the calculation method to be used. If FORM or SORM is implied, the intersection of $E_1(\mathbf{X})$ and $E_2(\mathbf{X})$ must be an intersection of single events.

In structural reliability a conditional event is used to model *inspection no-find* situations (no equality events) and together with the measured value variable (involves equality events) to model *inspection find* situations.

A conditional event is shown in Figure 3.4.

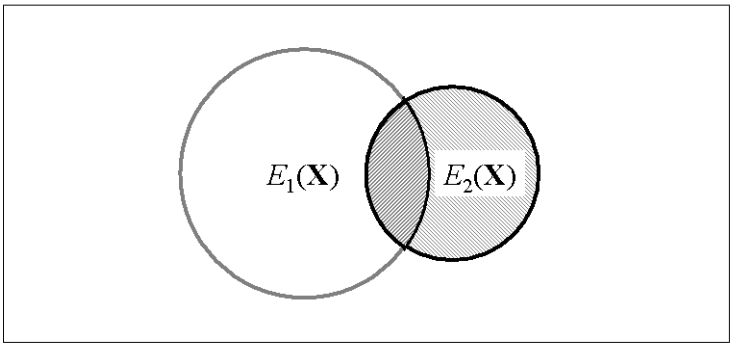


Figure 3.4 Conditional Event

4 MODEL INFORMATION OBTAINED BY USE OF PROBAN

4.1 General

This section describes the type of information about the model that Proban delivers:

- The event *probability*.
- The corresponding *reliability index*.
- The event *in-crossing rate*.
- The *first passage probability into the event.*
- The *cumulative probability distribution* of a random variable.
- The *density distribution* of a random variable.
- The *derivative of the probability* with respect to one or more measured values and/or event threshold values.
- The *conditional probability* for an event conditioned on another event involving (a mixture of) inequality events, equality events and measured values.
- The *conditional distribution* of a variable conditioned on an event.
- A *parameter study* of a probability or crossing rate as function of a numeric parameter in the model.
- The *parametric sensitivity factor* of the probability or the crossing rate with respect to a numeric parameter in the model.
- The *parametric sensitivity factors* of the four first statistical moments of the distribution of a random variable.

- The *uncertainty importance factors* which measures the importance to the probability or crossing rate of modelling a random variable as random rather than as a numerical value, the distribution median being the numerical value.

A number of calculation methods are available

- The FORM-*approximation*, which approximates the boundary of an event by linear forms in the *u*-space.
- The SORM-*approximation*, which approximates the boundary of an event by linear and quadratic forms in the *u*-space.
- The *Monte Carlo* simulation method which samples points according to the distribution of the model variables.
- The *Directional* simulation method which samples directions in the *u*-space, uniformly distributed on the surface of the *n*-dimensional unit hypersphere located at the origin, *n* being the dimension of the *u*-space.
- The *Axis-Orthogonal* simulation method, which first finds the design point in the *u*-space and subsequently samples directions parallel to the *u*-space axis of the intersection event or single event calculated on.
- The *Designpoint* simulation method, which first finds the design point in the *u*-space and subsequently employs the Monte Carlo simulation method in the *u*-space but centred at the design point.
- The *Latin-Hypercube* simulation method which employs a strategy for stratified sampling of the distribution of a random variable.
- The *Adjusted* simulation method, which permits a Monte Carlo simulation to be centred at a user defined point and for variables to sampled according to a stratified method
- The *Mean-Value* FORM method which estimates the distribution of a random variable assuming that the *u*-space gradient changes little in the probability interval of interest.

4.2 Event Probability

The event probability, P_E, is the probability that an outcome of the stochastic process X yields the event E,

$$P_E = P(E(\mathbf{X})) \tag{4.1}$$

The probability calculations in Proban are formulated in terms of a set of independent standard normal random variables **U**. To facilitate the computations, the model variables **X**, as defined by the user, are mapped onto an equivalent model in terms of **U**. (The mapping **X**=**T**(**U**) is a computational trick. The user does not have to deal with this.) The mapping is outlined in Chapter 6. The variable space defined by U_i, i=1,2,...,n is denoted as the *u-space*.

In Proban an event probability is estimated through the use of

- *Exact* methods: Exact results for the single event probability is obtained if the event boundary is linear or quadratic in u-space.

- *Approximation* methods: An approximation to the event is obtained by approximating the event boundary in *u*-space around a number of approximation points by linear, piecewise linear and quadratic surfaces. The choice of approximation points depends on the event in question and on the computational method used. The purpose is that the probability of the approximation can be calculated using exact methods, methods known to give good approximations and methods which bounds the probability. A First Order Reliability Method, FORM, makes use of linear and piecewise linear approximations to the event boundary. A Second Order Reliability Method, SORM, makes use also of quadratic approximations to the event boundary.

- *Simulation* methods: Estimates of the event probability, $\hat{P}_E(N)$, are obtained by use of simulation methods. The number of samples in the estimate is *N*. An unbiased sampling strategy has the property that $\hat{P}_E(N) \to P_E(N)$ as $N \to \infty$. In Proban, two unbiased sampling strategies are implemented, the Monte carlo simulation method and the Directional simulation method. In addition, two importance sampling methods, the Axis-Orthogonal simulation method and the Designpoint simulation method are implemented.

4.3 Reliability Index

The reliability index, β_R, is defined as the argument of the standard normal distribution which yields 1–the event probability,

$$\beta_R = \Phi^{-1}(1 - P_E) = -\Phi^{-1}(P_E) \tag{4.2}$$

It is customary to distinguish between

- The FORM *reliability index*, β_{FORM}, where P_E is obtained using a FORM approximation method.

- The SORM *reliability index*, β_{SORM}, where P_E is obtained using a SORM approximation method.

- The *generalized reliability index*, β_{R}, where P_E is the exact event probability.

While the event probability is most often a rapidly changing non-linear function of the distribution parameters, θ, the reliability index β_{R}, is often a more 'linear' function of θ, Figure 4.1. This implies that the parametric sensitivity factors for β_{R} are predictive in a wider range of θ then those for P_E.

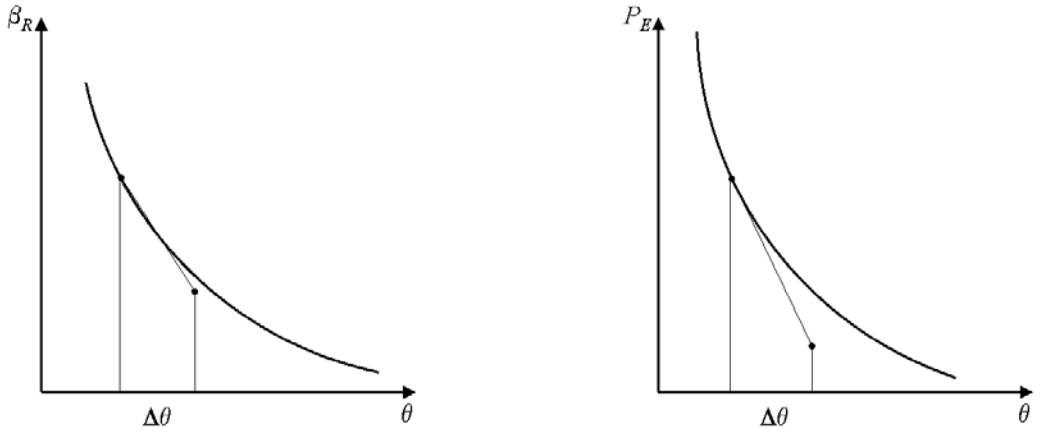


Figure 4.1 Derivative of Reliability vs Derivative of Probability

The other reason for using the reliability index is historically motivated. In early structural safety applications the safety was measured in terms of an index, Cornell (1969), Hasofer and Lind (1974), Ditlevsen (1979a), corresponding to β_R for special cases.

Proban permits the probability to be calculated in terms of the probability, the log10 of the probability and the reliability index.

4.4 Distribution of Variable, Simulation

The cumulative distribution of (the value of) the variable $Y(\mathbf{X})$ is

$$F_Y(y) = P(Y(\mathbf{X}) \le y) \tag{4.3}$$

and the density distribution is the corresponding derivative $f_Y(y)$.

Proban calculates the distribution of a variable by use of *simulation methods*. The simulation methods available are a) the *Monte Carlo* simulation method which samples points according to the distribution of the random variables in the model, and b) the *Latin-Hypercube* simulation method which is a stratified sampling strategy where each coordinate axis is divided into a prescribed number, *m*, of intervals of equal probability. The *m* points of each latin hypercube are sampled using one point from each axis interval only once.

During sampling, the four first statistical moments of the distribution, i.e. the *mean*, μ, the *standard deviation*, σ, the *skewness*, δ, and the *kurtosis*, κ, are estimated. (The kurtosis proper is used here. Kurtosis excess is obtained from kurtosis proper by subtracting 3.) Using *N* points in the sample, the unbiased estimator for the mean is

$$\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} g(\mathbf{u}_i) \tag{4.4}$$

The unbiased estimator for the standard deviation is

$$\hat{\sigma} = \left(\frac{1}{N-1} \sum_{i=1}^{N} (g(\mathbf{u}_i) - \hat{\mu})^2 \right)^{1/2} \tag{4.5}$$

The unbiased estimator for the skewness is

$$\hat{\delta} = \frac{\frac{N}{(N-1)(N-2)} \sum_{i=1}^{N} (g(\mathbf{u}_i) - \hat{\mu})^3}{\hat{\sigma}^3} \tag{4.6}$$

The unbiased estimator for the kurtosis (proper) is

$$\hat{\kappa} = \frac{\frac{N(N+1)}{(N-1)N-2)(N-3)} \sum_{i=1}^{N} (g(\mathbf{u}_i) - \hat{\mu})^4}{\hat{\sigma}^4} - \frac{3(3N-5)}{(N-2)(N-3)} \tag{4.7}$$

4.5 Distribution of a Variable, FORM/SORM Methods

The probability distribution of the variable $Y(\mathbf{X})$ is

$$F_G(\theta) = P(Y(\mathbf{X}) \leq \theta) \tag{4.8}$$

The parameter study on $F_G(\theta)$ with θ as parameter is executed by user defining the variable $G(\mathbf{X})=Y(\mathbf{X})-\theta$, user creating the event $E=G(\mathbf{X}) \leq 0$ and user executing a parameter study for θ. Both FORM, SORM and probability simulation methods may be used. Subsequently one may fit a distribution to the result of the parameter study to obtain the density distribution $f_G(\theta)$ as well.

The Mean Value FORM is a parameter study on $F_G(\theta)$ with θ as parameter and is executed by Proban defining the variable $G(\mathbf{X})=Y(\mathbf{X})-\theta$, Proban creating the event $E=G(\mathbf{X}) \leq 0$ and Proban executing a parameter study for θ. The assumption behind the Mean Value FORM method is that the direction to the approximation point of the θ-shifted function in terms of u-space variables, $g(\mathbf{u})-\theta$ is closely approximated by $\nabla g(\mathbf{0})$. Here $g(\mathbf{u}) = G(\mathbf{T}(\mathbf{U}))$. It is thus sufficient to compute the value of the event function at a number of points within prescribed probability limits in order to obtain an approximation to the distribution of the variable. Optionally the gradient $\nabla g(\mathbf{u}_l)$ where $\mathbf{u}_l$ corresponds to the lower probability bound, and the gradient $\nabla g(\mathbf{u}_u)$, where $\mathbf{u}_u$ corresponds to the upper probability bound, are used to modify the directions to the approximation points of the θ-shifted event.

4.6 Crossing Rate

The calculation of the crossing rate uses the parallel system formulation developed from Madsens formula, Hagen and Tvedt (1990). If there are m single event functions, G_i, i=1,2,...,m corresponding to the events E_i, then the crossing rate for the intersection event is

$$\nu(t) = \sum_{i=1}^{m} \frac{\partial}{\partial \theta_i} P\left(\bigcap_{j \neq i}^{m} G_j \leq 0 \bigcap G_i + \theta_i \dot{G}_i \leq 0 \bigcap -\dot{G}_i \leq 0 \right)_{\theta_i = 0} \tag{4.9}$$

Here $\dot{G}_i$ is the time derivative process of G_i.

Notice that the parallel system formulation works with any distribution model that can be mapped onto the *u*-space. However, the time derivative process must have zero mean since nonstationarity is handled only through explicit time dependency.

If the model includes variables not assigned time derivatives, then these are averaged over.

If the process is non-stationary, and the duration of the process is given, then the crossing rate is averaged over the duration. A trapezoidal quadrature is used to integrate over time.

A variable which is assigned as a time derivative, may itself be assigned a time derivative.

The starting time and the duration of the process are assigned to the process time variable.

The number of points in the quadrature is defaulted to 6. The value may be changed by use of the command DEFINE ANALYSIS OPTION TIME-DEPENDENT.

4.7 First Passage Probability

The first passage probability is calculated by use of the parallel system method for calculation of the crossing rate conditioned on the time-independent variables in the model and using the Poisson assumption for the crossing probability, Hagen and Tvedt (1990). The time-independent variables are integrated by using a formulation by Wen and Chen (1989). The Poisson assumption yields

$$P_{FPP} = 1 - (1 - P(T_0))^m \exp\left(\frac{-mn \int_{T_0}^{T_0 + D} \nu(t)dt}{(1 - P(T_0))} \right) \tag{4.10}$$

in which the crossing rate $\nu(t)$ calculated by the parallel system method while $P(T_0)$ is the probability of being in the failure set at the starting time T_0. The duration of the process, D, is measured from the starting time. A trapezoidal quadrature is used for the integration over time. The constant m permits an extreme value formulation in terms of the minimum of m independent realizations of the process. The constant n allows for the duration of the process to be split into n equal intervals (if periodical).

The default value of both m and n is 1. The default number of points in the quadrature is 6. The value may be changed by use of the command DEFINE ANALYSIS OPTION TIME-DEPENDENT.

4.8 Derivative of Probability

The partial derivative of the event probability with respect to one or more equality event threshold values or measurement values permits both the calculation of the density distribution of a variable and a separate calculation of the nominator and the denominator of a conditional probability calculation.

If the event E involves the single equality event, SE, as $E = \tilde{E} \bigcap \{\mathbf{x}; G_{SE}(\mathbf{x}) + \theta = 0\}$ then

$$\frac{\partial P_E}{\partial \theta} = \frac{\partial}{\partial \theta} P(\tilde{E} \bigcap G_{SE}(\mathbf{x}) + \theta \le 0)_{\theta = 0} \tag{4.11}$$

where θ is an auxiliary variable added to the equality event function.

If a *measured value* is defined for the event *SE*, then

$$\frac{\partial P_E}{\partial \theta} = \frac{\partial}{\partial \theta} P(\tilde{E} \bigcap G_{SE}(\mathbf{x}, \theta) \le 0)_{\theta = 0} \tag{4.12}$$

where θ is an auxiliary variable added to the measured value.

Proban checks the definition of the system at hand for single equality events and for measured values and sets up the appropriate formulation.

Note that the partial derivative of Equation (4.11) or of Equation (4.12) can be simulated if only one single equality event is included. This is done by simulating the parametric sensitivity factor with respect to θ.

4.9 Conditional Probability

In many cases the interest is on computing the probability that an event occur conditioned on the occurrence of another event, e.g., the event that a structural element fails within a specified number of years conditioned on the non-detection of cracks or the detection of a crack of a measured size, Madsen (1987).

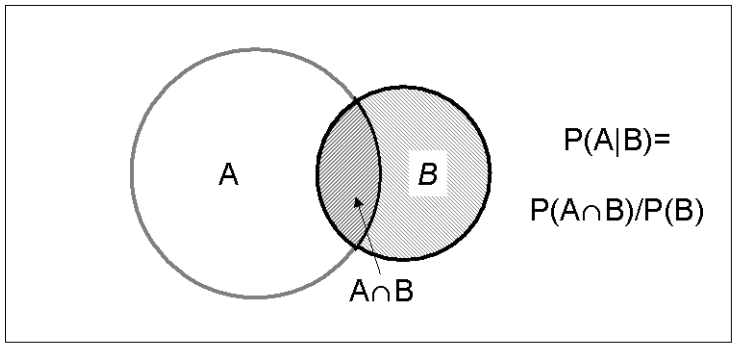


Figure 4.2 Law of Conditional Probability

If the problem at hand includes no single equality events, then the law of conditional probability gives:

$$P_{E_A|E_B} = \frac{E_A \bigcap E_B}{P_{E_B}} \tag{4.13}$$

If the event E_B includes the single equality event, *SE*, as $E_B = E_{\tilde{B}} \bigcap \{\mathbf{x}; G_{SE}(\mathbf{x}) = 0\}$ then

$$P_{E_A|E_B} = \frac{\frac{\partial}{\partial\theta} P(E_A \bigcap E_{\tilde{B}} \bigcap G_{SE}(\mathbf{x}) + \theta \le 0)_{\theta = 0}}{\frac{\partial}{\partial\theta} P(E_{\tilde{B}} \bigcap G_{SE}(\mathbf{x}) + \theta \le 0)_{\theta = 0}} \tag{4.14}$$

where θ is an auxiliary variable added to the single equality event function.

If a *measured value* is defined for the event *SE*, then

$$P_{E_A|E_B} = \frac{\frac{\partial}{\partial\theta} P(E_A \bigcap E_{\tilde{B}} \bigcap G_{SE}(\mathbf{x};\theta) \le 0)_{\theta = 0}}{\frac{\partial}{\partial\theta} P(E_{\tilde{B}} \bigcap G_{SE}(\mathbf{x};\theta) \le 0)_{\theta = 0}} \tag{4.15}$$

where θ is an auxiliary variable added to the measured value.

Proban checks the definition of the system at hand. If it is a conditional system, then Proban checks for single equality events and for measured values and sets up the appropriate formulation.

4.10 Conditional Distribution

The distribution of a variable $Y_E(\mathbf{X})$ conditioned on the event $E(\mathbf{X})$ is simulated by excluding samples outside the event.

$$Y_E(\mathbf{X}) = \{Y(\mathbf{X}); \mathbf{X} \in E\} \tag{4.16}$$

4.11 Parametric Sensitivity Factor for Probability, Crossing Rate and Derivative of Probability

The parametric sensitivity factor, f_θ, is defined as the derivative of the reliability measure with respect to the parameter θ. The parameter θ is either a variable of type Fixed or a numerical parameter of a function or of a distribution. Thus, if the main result of the calculation is the reliability index, then

$$\beta_\theta = \frac{\partial\beta_R}{\partial\theta} \tag{4.17}$$

and similarly for the probability *P* or log10(*P*). The parametric sensitivity factor can be used to estimate the change of the reliability index due to a change Δθ of the parameter:

$$\beta_R(\theta + \Delta\theta) = \beta_R(\theta) + \beta_\theta \Delta\theta \tag{4.18}$$

The method used for the computation of the parametric sensitivity factors depends on the calculation method for the main result. If FORM or SORM is used, then an asymptotic result and a result consistent

with the FORM approximation are available. If the *Directional simulation* methods is used, then the parametric sensitivity factors are simulated simultaneously, see Chapter 21.

A subset of the parameter is selected for sensitivity calculation by use of the command ASSIGN SENSITIVITY. The computation of parametric sensitivity factors is optional. The command DEFINE ANALYSIS-OPTION SENSTIVITY allows the user the options of none, all or selected parameters for parametric sensitivity calculation.

4.12 Parametric Sensitivity Factor for Distribution

The parametric sensitivity factor for the four first statistical moments of the distribution of a variable with respect to the parameter θ are defined as the derivatives of the statistical moments with respect to the parameter θ. The parameter θ is either a variable of type Fixed or a numerical parameter of a function or of a distribution. Thus, the definition of the parametric sensitivity factor of the *mean* of the distribution is:

$$\mu_\theta = \frac{\partial \mu}{\partial \theta} \tag{4.19}$$

For the *standard deviation* of the distribution the definition is

$$\sigma_\theta = \frac{\partial \sigma}{\partial \theta} \tag{4.20}$$

For the *skewness* of the distribution the definition is

$$\delta_\theta = \frac{\partial \delta}{\partial \theta} \tag{4.21}$$

For the *kurtosis* of the distribution the definition is

$$\kappa_\theta = \frac{\partial \kappa}{\partial \theta} \tag{4.22}$$

The parametric sensitivity factors are computed during the sampling. The sensitivity factors are the derivatives of the sample estimates $\hat{\mu}$, $\hat{\sigma}$, $\hat{\delta}$ and $\hat{\kappa}$.

A subset of the parameter is selected for sensitivity calculation by use of the command ASSIGN SENSITIVITY. The computation of parametric sensitivity factors is optional. The command DEFINE ANALYSIS-OPTION SENSTIVITY allows the user the options of none, all or selected parameters for parametric sensitivity calculation.

4.13 Uncertainty Importance Factors

The uncertainty importance factor, a_μ, is defined as the derivative of the reliability index with respect to the mean μ of the corresponding u-space variable, Hohenbichler and Rackwitz (1986). Thus

$$\alpha_\mu = \frac{\partial \beta_R}{\partial \mu} \tag{4.23}$$

The uncertainty importance factor indicates the importance of modelling the random variable X as a distributed variable rather than as a fixed valued variable, the median of the distribution being the fixed value.

The uncertainty importance factor are computed only if the FORM or SORM is involved or if the Directional simulation method is employed.

In the sequel, let $\mathbf{a}_\mu$ be a u-space unit vector. In the case of a single event, the vector $\mathbf{a}_\mu$ is simply

$$\mathbf{a}_\mu = \text{sgn}(g(\mathbf{0}))\mathbf{a} \tag{4.24}$$

where $\mathbf{a}$ is the direction vector to the design point, see Chapter 6. If the random variables $\mathbf{X}$ are independent, an estimate of the reliability index obtained by representing X_i by its median, is given by the *omission* factor, Madsen (1988a),

$$\omega = (1 - a_{\mu,i}^2)^{1/2} \tag{4.25}$$

The new reliability index is

$$\beta_{R,\text{new}} = \omega \beta_{R,\text{old}} \tag{4.26}$$

If the random variable X_i is one out of a number m mutually independent random variables, the omission factor resulting from representing the random variables X_i, i=1,2,...,m by their successive median, is

$$\omega = \left(1 - \sum_{i=1}^{m} a_{\mu,i}^2 \right)^{1/2} \tag{4.27}$$

In Proban, for an independent variable, $100 a_i^2$ is printed. For a group of mutually dependent variables, $100 \sum_{i=1}^{m} a_{\mu,i}^2$ is printed. These numbers can be interpreted as the percentage of the total uncertainty, reflected in the reliability index β_R, due to the corresponding random variable or group of random variables.

In the case of compound events the uncertainty importance factors printed by Proban are the same percentages of the reliability index as in the case of a single event. However, the interpretation of the factors as omission factors, Equation (4.27), may not apply.

5 GUIDE TO METHODS

5.1 Introduction

A variety of methods are made available with Proban. The application of a particular method depends, as in all mathematical modelling and analysis, on the problem at hand and on the objective of the analysis. In the following some important characteristics of the methods are tabulated to assist the user to choose a method appropriate to the problem at hand. Important to the selection of a method is:

- The objective of the analysis.
- The number of random variables involved.
- The computational cost of evaluating the event function.
- The properties of the event unction, e.g. existence, continuity and differentiability.
- The reliability level of interest.
- The probability range of interest for a variable or an event function.

The guide is divided into

5.2 Event Probability

5.2.1 FORM

The First Order Reliability Method linearizes the single event boundaries at a selected set of u-space points. The applicability of the method is as follows:

- *Systems*: The method is available to SE, USE, ISE and UISE systems
- *Event function*: The event function must be continuously differentiable.

- *Results*: Probability, parametric sensitivity factors, and uncertainty importance factors.

- *Applicability*: The method is particularly useful for the high reliability problems often encountered in engineering. The method assumes that each single event boundary is linear or nearly linear in the neighbourhood of each approximation point. The method is also useful if qualitative statements about the model are sufficient, e.g., if the results are used to compare models.

- *Check*: If the applicability of FORM to a specific model is uncertain, then the FORM result should be compared for selected cases with the result obtained by using simulation. Often comparison with the SORM result will suffice.

- *Computation cost*: The cost of a FORM calculation is essential the cost of solving a number of optimization problems, one for each subsystem in the model. Assume that the system is composed of m subsystems, that subsystem no. i is composed of k_i single events and the single event no. j of subsystem no. i have n_{ij} random variables. If solving the optimization problem for subsystem no. i requires r_i iterations, then the computational effort in terms of function evaluations is of the order

$$\sum_{i=1}^{m} r_i \sum_{j=1}^{k_i} (n_{ij} + 1) \tag{5.1}$$

5.2.2 SORM - SINGLE EVENT

In case of a single event, the Second Order Reliability Method, SORM, fits a quadratic surface to the boundary of the event. Three types of quadratic fits are available, the parabolic fit, the diagonal fit and a fit based on the Taylor expansion to the second order of the event function. The two first are invariant with respect to the formulation of the event function, for as long as the event boundary is preserved, while this is not true for the fit based on the second order Taylor expansion. The applicability of the method is as follows:

- *Event function*: The event function must be twice continuously differentiable.

- *Results*: Probability, parametric sensitivity factors, and uncertainty importance factors.

- *Applicability*: The method assumes that the event boundary is closely approximated by a quadratic surface in the neighbourhood of the approximation point. The invariance property of the parabolic form makes it generally applicable. The diagonal fit is generally applicable if the event function can be written as $G(\mathbf{x}) = \sum_{i=1}^{n} h_i(x_i)$ and the random variables X_i are mutually independent. It is also applicable when the curvatures of a full parabolic approximation have small values. The lack of invariance implies that care has to be exercised whenever the Taylor expansion to the second order is used.

- *Check*: If the applicability of SORM to a specific model is uncertain, then the SORM result should, for selected cases be compared to the result obtained by using simulation.

- *Computation cost*: The cost of a SORM calculation is the cost of a FORM calculation plus the cost of deriving a quadratic approximation to the event boundary. If there are n random variables in the model, then the additional number of function evaluations is: a) Parabolic approximation: $n(n\text{-}1)/2$, b) Diagonal approximation: $2n$, c) Taylor expansion to the second order: $n(n+1)/2$. If n is large, say 100, then simulation methods become competitive also for small probabilities, say 10^{-3}. In the cases a) and c) the additional cost is 5000 function evaluations.

5.2.3 SORM - SYSTEM

In case the system comprises more than one single event, the parabolic SORM is used. The SORM is used to compute a correction factor to the FORM.

- *Systems*: The method is applicable to USE, ISE and UISE systems.
- *Event function*: The event function must be twice continuously differentiable.
- *Results*: Probability, parametric sensitivity factors, and uncertainty importance factors.
- *Applicability*: Small intersection (the median of the distribution, **u**=**0** in *u*-space, is not in the event): The method assumes that the joint intersection of the boundaries of active single events is closely approximated by a parabolic surface in the neighbourhood of the approximation point.

  Large intersection (**u**=**0** is in the event): The method assumes that each single event boundary is closely approximated by a parabolic surface in the neighbourhood of its approximation point.

  These properties are typical for the high reliability problems often encountered in engineering.
- *Check*: If the applicability of SORM to a specific model is uncertain, then the SORM result should, for selected cases be compared to the result obtained by using simulation.
- *Computation cost*: Small intersection: If there are n random variables and k active single events in the model, then the additional number of function evaluations is: $k(n\text{-}k)(n\text{-}k\text{-}1)/2$. Large intersection: Number of single events × cost of SORM for single events.

5.2.4 Monte-Carlo Simulation

The Monte Carlo simulation method samples from the joint distribution of the random basic variables.

- *Systems*: All systems except systems involving equality events.
- *Event function*: The event function should be possible for all realizations of the basic variables, i.e., all the points where the joint density distribution has a positive value. If the event function cannot be calculated at some point **x**, then the point contributes zero to the probability.
- *Results*: Probability.
- *Applicability*: The Monte Carlo simulation method gives an unbiased estimate of the probability. The method is efficient for the simulation of the central part of the distribution, i.e., probabilities in the range, say [0.1,0.9].
- *Computation cost*: The number of function evaluations required to obtain a reasonable reliable estimate of the probability P_E for the un-adjusted Monte Carlo method is approximately $100/P_E$. The purpose of the adjusted method is to improve on the efficiency of the method by exploiting user knowledge about the domain of the event which contribute to the probability.

5.2.5 Directional Simulation

The Directional simulation method samples directions uniformly distributed on the *n*-dimensional unit sphere in *u*-space and accumulates a weighted sum of the event probabilities conditioned on the sampled directions.

- *Systems*: All systems except systems involving equality events.
- *Event function*: The calculation of the event function should be possible for all valid realizations of the basic random variables.
- *Results*: Probability, parametric sensitivity factors and uncertainty factors.
- *Applicability*: The Directional simulation method gives an unbiased estimate of the probability. The method is efficient for simulation of small probabilities, i.e., in the ranges (0,0.1] and [0.9,1].
- *Computation cost*: The computational cost is proportional to the area of the surface of the *n*-dimensional unit sphere that generates directions intersecting the event. As this proportion tends to be reduced with increase in the number of random variables, the Monte Carlo method becomes competitive if the number of random variables is large. The Monte Carlo method should be used for *n* large, say $n \geq 100$. The area mentioned is also reduced for small intersection systems. For such systems the axis-orthogonal simulation method or the design point simulation method may be better suited.

5.2.6 Axis-Orthogonal Simulation

The Axis-Orthogonal simulation method is an importance sampling method that simulates a correction to the FORM approximation.

- *Systems*: SE and ISE (small intersection).
- *Event function*: A FORM approximation must be possible.
- *Results*: Probability, parametric sensitivity factors and uncertainty importance factors.
- *Applicability*: The Axis-Orthogonal simulation method gives an unbiased estimate of the probability in the case there is one local design point only. The method is recommended for The method is recommended for simulation of small probabilities, i.e., in the ranges (0,0.1] and [0.9,1].
- *Computation cost*: The computational cost is the cost of FORM for the same problem + cost of the additional sampling. This additional cost depends on how well the event boundary of the SE or ISE system is approximated by linear surfaces. If well approximated, the number of additional samples may be down to 50.

5.2.7 Designpoint Simulation

The Designpoint simulation method is an importance sampling method that simulates the probability around the design point (closest to u-space origin approximation point) by using a u-space Monte Carlo simulation method with the mean of the u-space samples adjusted to the design point.

- *Systems*: SE and ISE (small intersection).

- *Event function*: Normally the same as for FORM. However, by using the RFCRC optimization method, continuous differentiability at the design point is not required.
- *Results*: Probability, parametric sensitivity factors and uncertainty importance factors.
- *Applicability*: The Designpoint simulation method gives an unbiased estimate of the probability in the case there is one local design point only. The method is useful for simulation of small probabilities, i.e., in the ranges (0,0.1] and [0.9,1] and for event boundaries that are not well approximated by linear or quadratic surfaces in the vicinity of the design point.
- *Computation cost*: The computational cost is the cost of FORM for the same problem + cost of the additional sampling.

5.3 Conditional Probability

The conditional event $E_C(\mathbf{X}) = (E_1|E_2)(\mathbf{X})$ makes direct calculation of conditional probabilities possible. It is the method to be used for calculating the probability of an event conditioned on a measured value.

- *Systems*: Restricted by the choice of calculation method.
- *Methods*: FORM, SORM and probability simulation methods are available for calculation of conditional probabilities.
- *Results*: Probability, parametric sensitivity factors and uncertainty importance factors.
- *Applicability*: As for the probability calculation method actually used.
- *Computation cost*: As for the probability calculation method actually used. Notice that two probability calculations are carried out, one for $(E_1 \cap E_2)(\mathbf{X})$ and one for $E_2(\mathbf{X})$.

5.4 Crossing Rate

The rate of a stochastic process crossing into an event is calculated by using FORM or SORM in conjunction with the parallel system method. However, the present Proban implementation permits SORM only for a model that has no time-independent variables.

- *Systems*: The crossing rate can be calculated for SE, USE, ISE and UISE systems. Equality events are not permitted.
- *Event function*: The event functions must be continuously differentiable (FORM) or twice continuously differentiable (SORM).
- *Results*: Crossing rate, parametric sensitivity factors and uncertainty importance factors.
- *Applicability*: Requirements as for FORM and SORM, dependent on which method is used.
- *Computation cost*: The cost of a crossing rate calculation depends on the problem at hand and the method used. If there are no time-independent variables, the cost is as for a single execution of a FORM or a SORM. If time-independent variables are present, then those variables are averaged over by a nested

reliability analysis. The outer loop is a FORM for a single event while the inner loop is a FORM for each execution of the inner loop. If the process is non-stationary, an integration over time takes place, executing a FORM or a SORM or a nested analysis for each time integration point.

5.5 First Passage Probability

The probability of a stochastic process being in or crossing into an event is calculated by using FORM or SORM in conjunction with the parallel system method. However, the present Proban implementation permits SORM only for a model that has no time-independent variables.

- *Systems*: The crossing rate can be calculated for SE, USE, ISE and UISE systems. Equality events are not permitted.
- *Event function*: The event functions must be continuously differentiable (FORM) or twice continuously differentiable (SORM).
- *Results*: Probability, parametric sensitivity factors and uncertainty importance factors.
- *Applicability*: Requirements as for FORM and SORM, dependent on which method is used.
- *Computation cost*: The cost of a first passage probability calculation depends on the problem at hand and the method used. If there are no time-independent variables, the cost is as for a single execution of a FORM or a SORM. If time-independent variables are present, then those variables are integrated over by a nested reliability analysis. The outer loop is a FORM for a single event while the inner loop is a FORM for each execution of the inner loop. If the process is non-stationary, an integration over time takes place, executing a FORM or a SORM or a nested analysis for each time integration point.

5.6 Distribution of Variable

5.6.1 Monte-Carlo Simulation

The Monte Carlo simulation method samples from the joint distribution of the random basic variables.

- *Variables*: All types of (vector)variables except probability and time variables.
- *Results*: Distribution, fitted distribution, parametric sensitivity for the four first central moments of the distribution. (Result can subsequently be fitted to a library distribution.)
- *Applicability*: The Monte carlo simulation method gives an unbiased estimate of the distribution. The method is efficient for the simulation of the central part of a distribution, i.e., probabilities in the range, say [0.01,0.99].

5.6.2 Latin-Hypercube Simulation

The Monte Carlo simulation method uses stratified sampling.

- *Variables*: All types of (vector)variables except probability and time variables.

- *Results*: Distribution, fitted distribution, parametric sensitivity for the four first central moments of the distribution. (Result can subsequently be fitted to a library distribution.)

- *Applicability*: The Latin-Hypercube simulation method gives an unbiased estimate of the distribution. The method is efficient for the simulation of the central part of a distribution, i.e., probabilities in the range, say [0.01,0.99]. As compared to the Monte Carlo simulation method, the Latin Hypercube simulation method yields typically a more precise estimate of the mean and the standard deviation of the distribution. The method is particular efficient if the value of the random variable Y(**X**) is monotone in each of the basic variables **X**.

5.6.3 Mean-Value FORM

If the calculation of a variable is costly, then simulation methods may be prohibitively costly for obtaining the distribution of a random variable. In such cases the Mean-Value FORM may be used if the gradient of the variable, $\nabla y(\mathbf{u})$, is predictive for the function values in the probability range of interest. (This is often true for event functions defined for structural reliability applications.)

- *Variables*: All types of variables except probability and time variables.

- *Results*: Distribution. (Result can subsequently be fitted to a library distribution.)

- *Applicability*: The central part of the distribution, dependent on the linearity of the function.

- *Computation cost*: Up to three gradient evaluations and 16 additional function evaluations.

5.6.4 Distribution of Variable from Parameter Study

By subtracting an auxiliary parameter, θ, from the variable,

$$y(\mathbf{x}) - \theta \leq 0 \tag{5.2}$$

and performing a parameter study to establish $P(\theta)$, then the cumulative distribution for θ is established. The result may subsequently be fitted to a distribution in order to obtain also the density distribution for the variable.

- *Variables*: All types of variables except time variables.

- *Results*: Distribution. (Result can subsequently be fitted to a library distribution.)

- *Applicability*: The whole or the tail part of the distribution, dependent on the linearity of the function and the calculation method used.

- *Computation cost*: Depends on the probability calculation method used.

5.7 Parametric Sensitivity for Probability, Crossing Rate and Derivative of Probability

The parametric sensitivity factor is the derivative of the main result with respect to a parameter in the model.

The parametric sensitivity factors are computed using Directional simulation or a method involving FORM.

If FORM is involved, then the sensitivity factors are assembled as a weighted sum of the subsystem sensitivity factors.

The methods available for an intersection of single events are the *analytic* method and the *asymptotic* method. If FORM is used, the analytic method yields a result that is derivative of the FORM approximation. The asymptotic method yields a result in accordance with an asymptotic formula, see Chapter 21. If SORM or a FORM based simulation method is used, then the results for FORM are scaled with a relation of the calculated result to the FORM result.

The analytic sensitivity factors are particularly useful for nested analysis, i.e., with variables of type probability, since this permits the use of gradient based optimization methods for finding the approximation points.

The analytic sensitivity factors are default.

5.8 Parametric Sensitivity Factor for Distribution Moments

The parametric sensitivity factor is calculated for the mean, the standard deviation, the skewness and the kurtosis of the variable.

The parametric sensitivity factors are assembled during simulation, and are thus consistent with the simulated results for the distribution.

The calculation of the sensitivity factors is about as costly per parameter as the computation of the general result. If sensitivity factors are calculated for n parameters, then the extra computational effort is about n times the calculation of the general result. It is therefore recommended to select a subset of the parameters for parametric sensitivity calculations.

5.9 Uncertainty Importance Factors

The method used for calculation of uncertainty importance factors is consistent selected for computation of the parametric sensitivity factors.

However, further evaluations of the event function are not required and the uncertainty importance factors are therefore inexpensive.

6 TRANSFORMATION OF VARIABLES

6.1 Motivation

Basic to the computational methods in use in Proban is the mapping of the probabilistic model in terms of n random basic variables onto an equivalent model in terms of n independent standard normal random variables. The mapping preserves all the statistical properties of the model. Denoting the random variables $\mathbf{X} = (X_1, X_2,..., X_n)$, the independent standard normal random variables $\mathbf{U}=(U_1, U_2,..., U_n)$ and the mapping function $\mathbf{T}$, the mapping of the random variables is

$$\mathbf{U} = \mathbf{T}(\mathbf{X}) \tag{6.1}$$

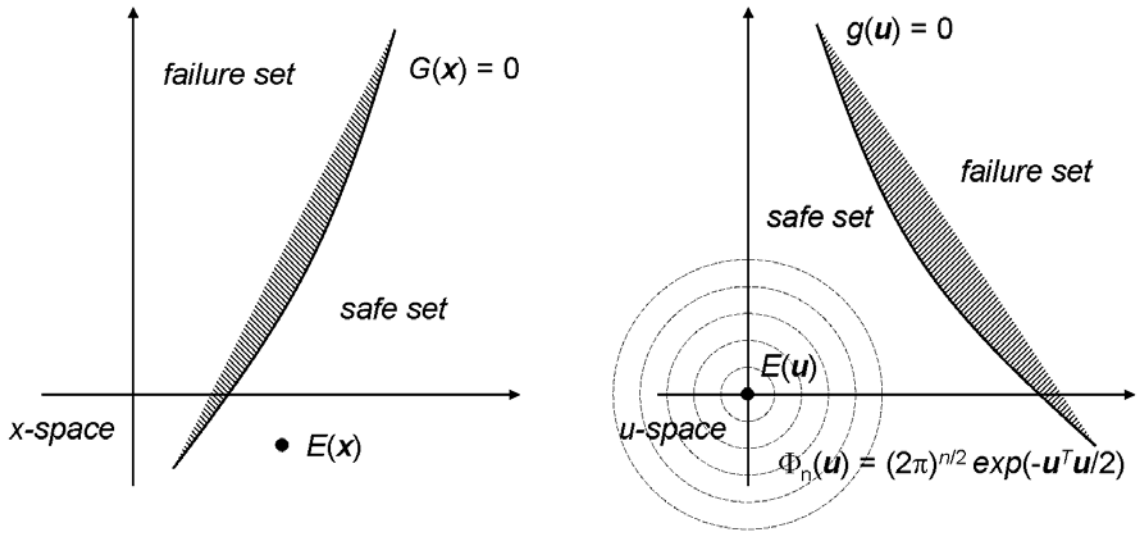


Figure 6.1 Mapping of the Event Function

The mapping of the event function is

$$g(\mathbf{u}) = G(\mathbf{T}^{-1}(\mathbf{u})) \tag{6.2}$$

An example of the mapping of a single event function is shown in Figure 6.1. It is customary to use the term *x*-space for the space of basic random variables and the term *u*-space for the space of independent standard normal random variables.

The usefulness of this mapping is due to

- *Rotational symmetry*: The probability content of domains in the *u*-space is invariant with the rotation of the domains about the origin **u**=**0**.

- *Exact results*: The probability can in *u*-space be obtained exactly for the linear domain, the parabolic domain, Tvedt (1988), Ref. /35/, and Tvedt (1989), Ref. /36/, the elliptic and hyperbolic domains, Rice (1980), Ref. /31/, and Helstrom (1983), Ref. /17/.

- *Approximations*: The probability can in *u*-space be obtained approximately for a convex domain bounded by linear surfaces, i.e. a domain defined by a parallel system of single events with linear event functions. The method used is the approximation method for the cumulative multinormal distribution, Hohenbichler (1984), Ref. /20/, and Gollwitzer and Rackwitz (1988), Ref. /12/.

If the intersection of a set of single events yields a parabolic surface in direction orthogonal to the subspace spanned by the gradients of the active constraints, then the probability content of the corresponding domain is approximated by a split of the linear parallel system and the parabolic domain, Hohenbichler (1984), Ref. /20/.

- *Asymptotic justification*: Breitung's theorem, Breitung (1984), Ref. /4/, states that the parabolic approximation to the single event obtained by fitting curvatures at the *u*-space design point, **u***, (the approximation point closest to the origin), asymptotically, i.e. as $\beta=|\mathbf{u}^*|\rightarrow\infty$, yields the true probability of the event. A similar result is valid for a small intersection system, Hohenbichler (1984), Ref. /20/. The rapid decay of the standard normal distribution away from the origin implies that the asymptotic approximations are useful also for moderate β, say $\beta\geq 2$.

It follows from this that if an event can be approximated by one of the above mentioned domains, or a composition of such domains, then an approximation to the probability content of the domain can be derived.

- *Simulation methods*: The unbiased directional simulation method, Bjerager (1988), Ref. /2/, and the axis orthogonal importance sampling method, Schall et al. (1988), Ref. /33/, both require a formulation in *u*-space variables.

6.2 Rosenblatt Transformation

Rosenblatt (1952), Ref. /32/, demonstrated a transformation of a set of multivariate random variables onto a set of independent random variables, each random variable being uniformly distributed on the interval [0,1]. In Hohenbichler and Rackwitz (1981), Ref. /18/, this result is used to define a mapping from the set of basic random variables onto a set of independent standard normal random variables. The mapping assumes that the conditional marginal distributions can be obtained in the following sequence:

$$\Phi(U_i) = F_{X_i}(X_i|X_1,X_2,\ldots,X_n) \qquad i = 1,2,\ldots,n \tag{6.3}$$

It is customary to denote this mapping function the Rosenblatt transformation. An example for a single variable is shown in Figure 6.2.

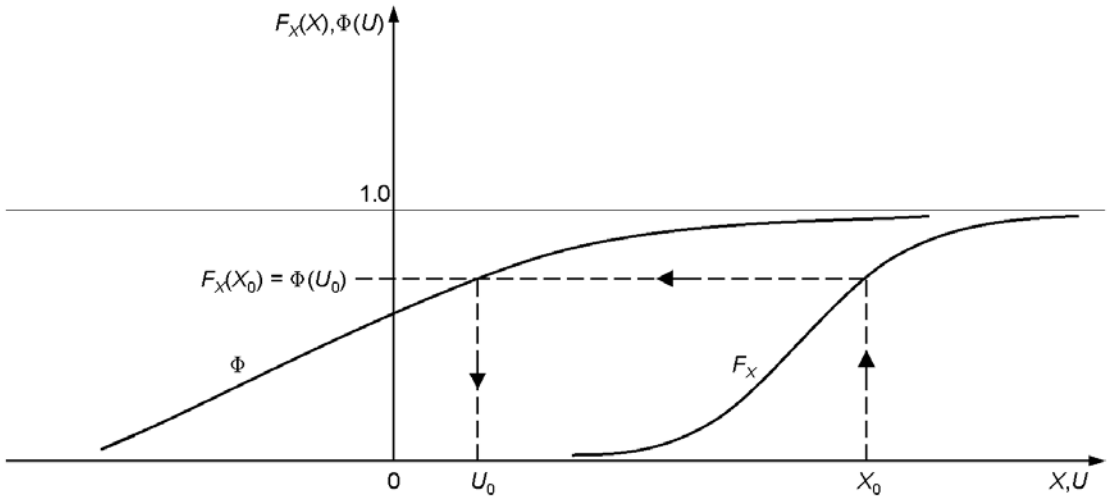


Figure 6.2 Rosenblatt Transformation for a Single Variable

For multivariate random variables the Rosenblatt transformation in general depends on the conditioning sequence, i.e. the numbering of the basic random variables (X_1, X_2,..., X_n). For simulation methods, the numbering is of little importance, since the mapping preserves all the statistical properties of the model. However, for a FORM or a SORM calculation this implies that the approximation of the event boundary in general depends on the numbering of the random variables. However, the effect of changing the numbering is generally small. (The asymptotic theory yields asymptotically the same result independent of the numbering.) In the following cases the transformation yields design points and results independent of the way the model is input to Proban:

- *Independent random variables*: In this case there is no conditioning on the right hand side of Equation (6.3).

- *Multinormal random variables*: In this case, the point of maximum likelihood on the event boundary in *x*-space corresponds to the point of maximum likelihood on the event boundary in *u*-space. Thus, the design point is independent of the numbering of the random basic variables.

- *Random variable correlated through the Nataf model*: In this case, the argument for the multinormal random variables applies to the *v*-space formulation of the function, see Appendix D. Because the *v*-space to *x*-space mapping is as for independent random variables, the statement is valid.

- *Random input parameters for distributions*: In this case, the natural modelling of the joint distribution yields a model in the form of the sequence assumed in Equation (6.3). Assuming the conditioning system of the model, the Rosenblatt transformation yields the same result independent of the numbering of the random basic variables.

7 FIRST ORDER RELIABILITY METHOD, FORM

7.1 Motivation

The First Order reliability method, FORM, applies to a probabilistic model formulated in terms of *u*-space variables. The basic idea is to approximate the event boundary by linear surfaces at a selected set of points. The computational methods available for linear and piecewise linear event boundaries are then applied to obtain an estimate (possibly in terms of bounds) of the event probability. The selection of linearization points depends on the system configuration and is outlined in the following subsections.

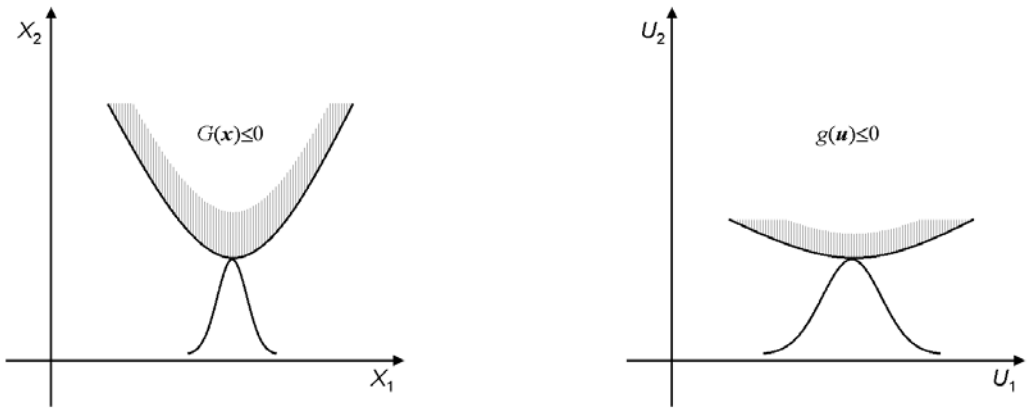


Figure 7.1 Mapping of Narrow Distributions

The motivation for using the FORM are:

- *Almost linear event boundary*: It is often so, e.g. in engineering applications, that the distributions used to model random variables are narrow, i.e., the coefficient of variation have a small value. In such cases the event boundary in the *u*-space is often significantly less curved than the event boundary in the *x*-space, see Figure 7.1. Because of this, results can be obtained by the FORM with a quality sufficient for the intended application.

- *Comparison of probabilistic models*: An interesting application of probabilistic methods is to compare probabilistic models, varying the model assumptions. In such cases the accuracy required is one that permits comparison rather than exact results.

- *Computational efficiency*: In many applications in engineering, the calculation of a variable may require computationally costly subroutines, e.g, the value of a variable may require use of the finite element method. In such cases, the use of other methods, e.g. simulation methods, may be computationally prohibitive for small failure probabilities.

- *Theoretical support*: Theoretical support for the FORM is offered by Breitung (1984), Ref. /4/, and Hohenbichler (1984), Ref. /20/. Their theorems state that the system reliability index obtained using the FORM, asymptotically, i.e. as the norm of the design point $|\mathbf{u}^*|$ tends to infinity, under certain restrictions on the curvatures, yields the true reliability index:

$$\beta_{\text{FORM}} \sim \beta_R, \qquad |\mathbf{u}^*| \to \infty \tag{7.1}$$

7.2 Single Event

The boundary of the single event is linearized at the point of maximum likelihood in *u*-space, the design point **u***, Figure 7.2:

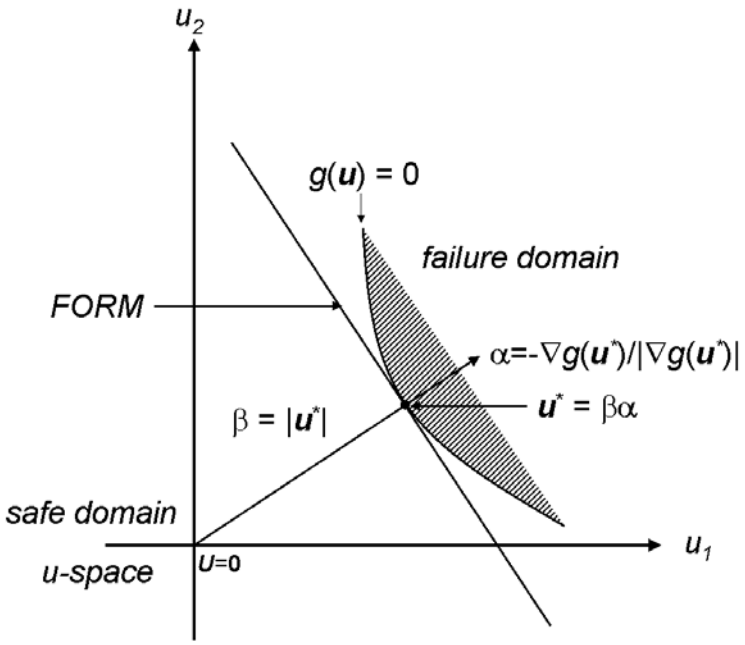


Figure 7.2 Linearization of Single Event Boundary

The FORM reliability index is:

$$\beta_{\text{FORM}} = |\mathbf{u}^*| \operatorname{sgn}(g(\mathbf{0})) \tag{7.2}$$

The corresponding probability of failure is:

$$P_{\text{FORM}} = \Phi^{-1}(\beta_{\text{FORM}}) \tag{7.3}$$

7.3 Intersection of Single Events

7.3.1 Linearization of Small Intersection

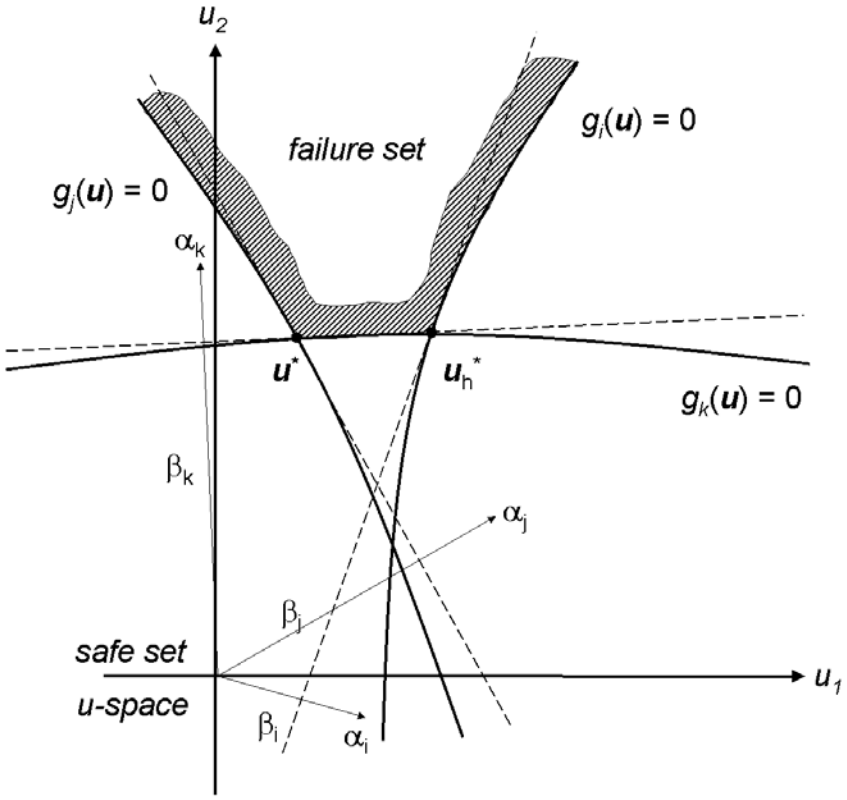


Figure 7.3 Linearization of Small Intersection

The term *small intersection* denotes an intersection of single events where the event function is non-negative at **u**=**0**, Figure 7.3. The steps in the linearization procedure are:

- *Active constraints*: The first linearization point is the point of maximum likelihood, the design point **u***, on the event boundary, see Figure 7.3. The single events which have zero valued functions at this point is denoted as *active*.

- *Inactive constraints*: The single events which are not active, are denoted as *inactive*. The inactive single events are linearized in succession, if possible. The linearization point of an inactive single event is the point of maximum likelihood on the part of its event boundary that intersects the event defined by the previous linearizations, see Figure 7.3. If this intersection is empty, then the event is deleted from further calculations.

Denoting the linearization of single event no. *i*:

$$g_i(\mathbf{u}) = \beta_i + \mathbf{a}_i^T \mathbf{u} \tag{7.4}$$

the linear surfaces defines a point, $-\mathbf{b}=(-\beta_i)$, on the cumulative multivariate normal distribution, with mean **0**, standard deviation **1** and correlation $\mathbf{R}=(\rho_{ij})=(\mathbf{a}_i^T \mathbf{a}_j)$. Thus

$$P_{\text{FORM}} = \Phi(-\mathbf{b};\mathbf{R}) \tag{7.5}$$

The corresponding reliability index is:

$$\beta_{\text{FORM}} = -\Phi^{-1}(P_{\text{FORM}}) \tag{7.6}$$

7.3.2 Linearization of Large Intersection

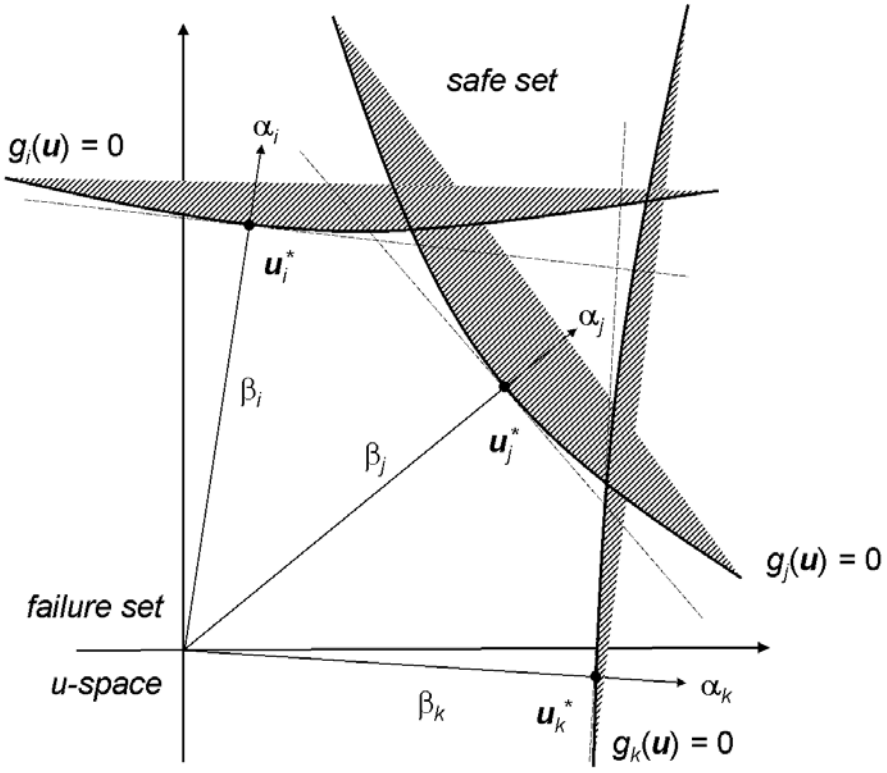


Figure 7.4 Linearization of Large Intersection

The term *large intersection* denotes an intersection of single events where the event function is strictly negative at $\mathbf{u}=\mathbf{0}$, Figure 7.4. The steps in the linearization procedure are:

- *Active constraints*: The single events are first linearized at their separate design points. A single event that has its linearization point on the event boundary of the linearized intersection is denoted as *active*, see Figure 7.4.

- *Inactive constraints*: The single events which are not active, are denoted as *inactive*. The inactive single events are linearized in succession, if possible. The linearization point of an inactive single event is the point of maximum likelihood on the part of its event boundary that intersects the event defined by the previous linearizations, see Figure 7.4. If this intersection is empty, then the event is deleted from further calculations.

Denoting the linearization of single event no. i:

$$g_i(\mathbf{u}) = \beta_i + \mathbf{a}_i^T \mathbf{u} \tag{7.7}$$

the linear surfaces defines a point, $-\mathbf{b}=(-\beta_i)$, on the cumulative multivariate normal distribution, with mean $\mathbf{0}$, standard deviation $\mathbf{1}$ and correlation $\mathbf{R}=(\rho_{ij})=(\mathbf{a}_i^T \mathbf{a}_j)$. Thus

$$P_{\text{FORM}} = 1-\Phi(-\mathbf{b};\mathbf{R}) \tag{7.8}$$

The corresponding reliability index is:

$$\beta_{\text{FORM}} = -\Phi^{-1}(P_{\text{FORM}}) \tag{7.9}$$

7.3.3 Proban Options

Proban detects whether the intersection is a small intersection or a large intersection. The user optionally chooses to exclude inactive constraints from the linearization procedure. The pertinent command is DEFINE FORM-SORM.

7.4 Union of Single Events

7.4.1 The Complementary Problem

Rather than computing a union of single events, one may choose to solve the complementary problem which is an intersection of single events. The relation of the probabilities is

$$P\left(\bigcup_{i=1}^{m} \{\mathbf{u};g_i(\mathbf{u}) \le 0\}\right) = 1 - P\left(\bigcap_{i=1}^{m} \{\mathbf{u};g_i(\mathbf{u}) \ge 0\}\right) \tag{7.10}$$

The intersection of single events is computed using the methods of Section 7.3.

7.4.2 Bounds

Using this method, each single event is linearized separately, Figure 7.5. An upper bound and a lower bound to the event probability of the linearized system is computed by making use of *simple bounds*, (first order bounds), see Appendix E 2 or *Ditlevsen bounds*, (second order bounds), see Appendix E 3. Denoting the

upper bound on the probability P_U and the lower bound P_L, the lower bound on the reliability index becomes:

$$\beta_{\text{FORM,L}} = -\Phi^{-1}(P_U) \tag{7.11}$$

The upper bound becomes:

$$\beta_{\text{SORM,L}} = -\Phi^{-1}(P_L) \tag{7.12}$$

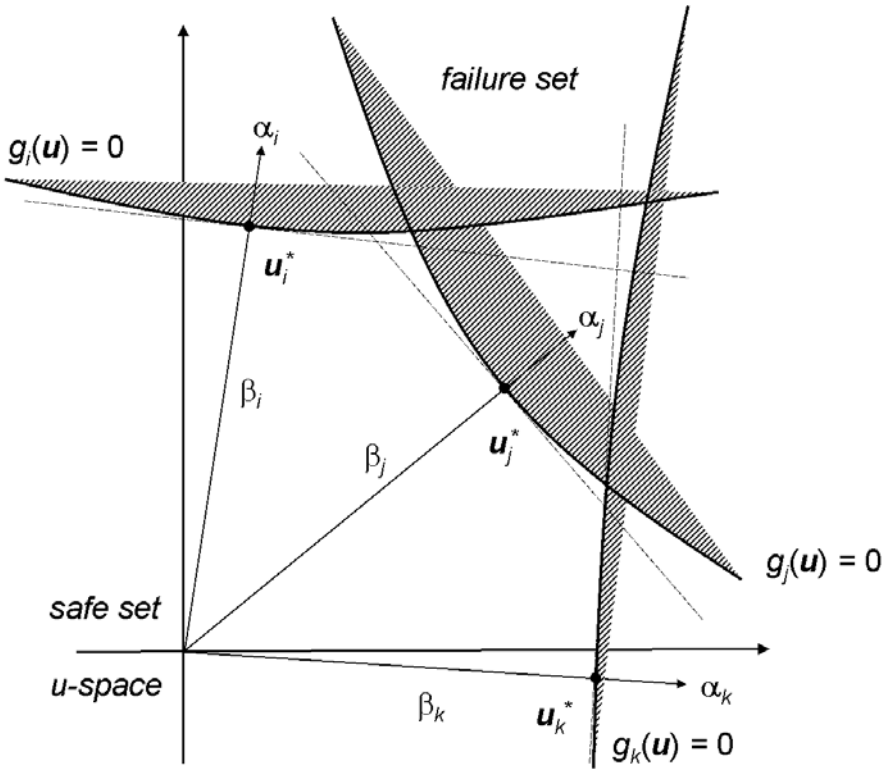


Figure 7.5 Linearization of Union of Single Events

7.4.3 Proban Options

In Proban solving the complementary problem is the default method for union of single events. The user optionally chooses a bounds method. The pertinent command is DEFINE FORM-SORM.

8 SECOND ORDER RELIABILITY METHOD, SORM

8.1 Motivation

The Second Order reliability method, SORM, applies to a probabilistic model formulated in terms of *u*-space variables. The basic idea is to approximate a single event boundary with a quadratic surface and an intersection of single events by a linear/parabolic surface. In each case the quadratic approximation is made at the design point. The computational methods available for quadratic forms and for linear and piecewise linear forms are then applied to obtain an estimate (possibly in terms of bounds) of the event probability.

The motivation for using the SORM are similar to those for using the FORM. It should, however, be noted that the SORM extends significantly the class of problems that can be treated by approximation methods because even highly curved event boundaries can be well approximated.

Theoretical support for the SORM is offered by Breitung (1984), Ref. /4/, and Hohenbichler (1984), Ref. /20/. Their theorems state that the event probability obtained using the SORM, asymptotically, i.e. as the norm of the design point $|\mathbf{u}^*|$ tends to infinity, under certain restrictions on the curvatures, yields the true event probability.

8.2 Single Event

Three methods are available for approximating the *u*-space boundary of a single event:

- The parabolic approximation, Figure 8.1, is a parabola fitted to the main curvatures of the boundary at the design point $\mathbf{u}^*$. The exact result for the probability of the parabolic domain is given in Tvedt (1988), Ref. /35/, and in Tvedt (1989), Ref. /36/. This result and the asymptotic result by Breitung (1984), Ref. /4/, and by Hohenbichler (unpublished) are implemented in Proban. See Appendix C for more detail.

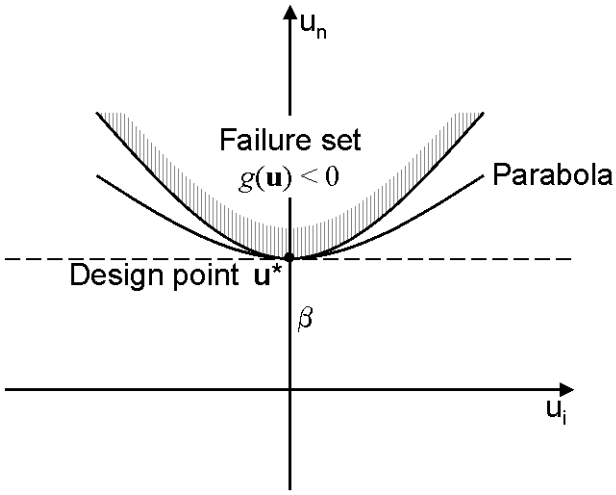


Figure 8.1 Parabolic Approximation

- The parabolic approximation is derived from the diagonal of **G**(**u***). **G**(**u***) is the matrix of second derivatives of the event function g(**u**) evaluated at the design point **u***. The SORM probability, P_{SORM}, is computed using the exact result for the parabolic quadratic form. The approximation is a modified version of a suggestion by Der Kiureghian et al. (1987), Ref. /8/.

- The approximation derived from the second order Taylor expansion of the event function at **u***, Figure 8.2. The exact result for a general quadratic form is given in Rice (1980), Ref. /31/, and Hellstrom (1989), Ref. /17/. See also Tvedt (1989), Ref. /36/.

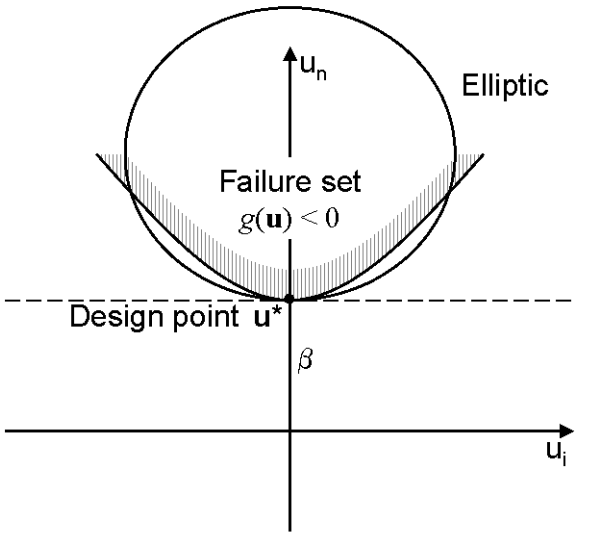


Figure 8.2 Second Order Taylor Approximation

The SORM reliability index is:

$$\beta_{SORM} = -\Phi^{-1}(P_{SORM}) \tag{8.1}$$

8.3 Intersection of Single Events

8.3.1 Approximation of Small Intersection

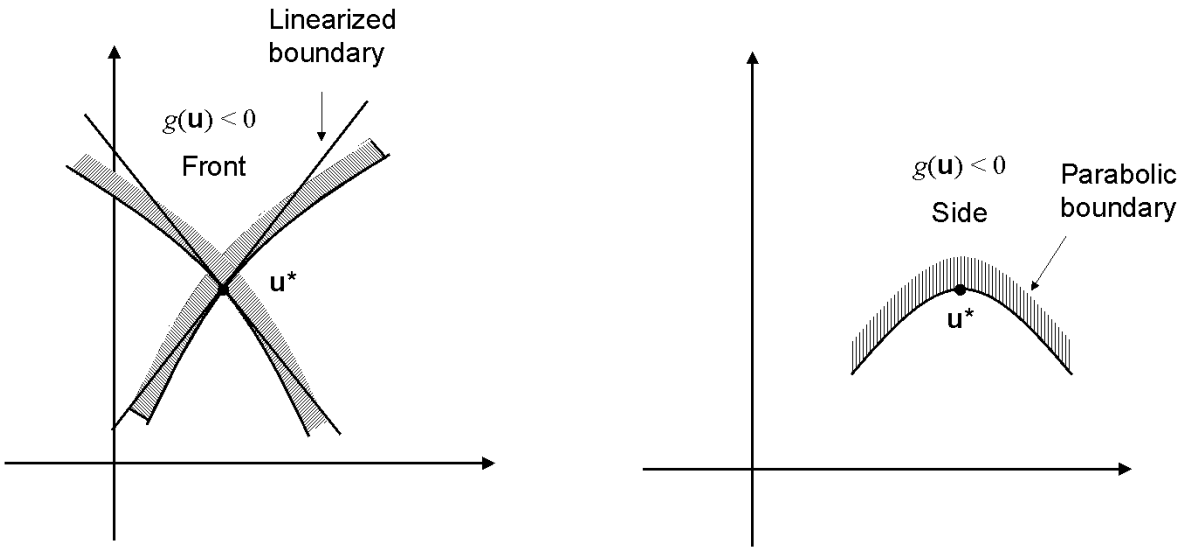


Figure 8.3 SORM Approximation of Small Intersection

The SORM approximation derives a correction factor to the FORM probability, Hohenbichler (1984), Ref. / 20/. The intersection of the single event boundaries yields a non-linear surface in the space orthogonal to the space spanned by the gradients at the design point, **u***, of the active constraints. This surface is approximated by a parabolic surface. The SORM probability of failure is in this case an asymptotic result for the linear/parabolic domain.

$$P_{SORM} = P_{FORM}\frac{Q_{SORM}}{\Phi(-|\mathbf{u}^*|)} \tag{8.2}$$

where Q_{SORM} is the separate probability of the parabolic form and $|\mathbf{u}^*|$ is the norm of the design point vector.

The method is illustrated in Figure 8.3.

8.3.2 Approximation of Large Intersection

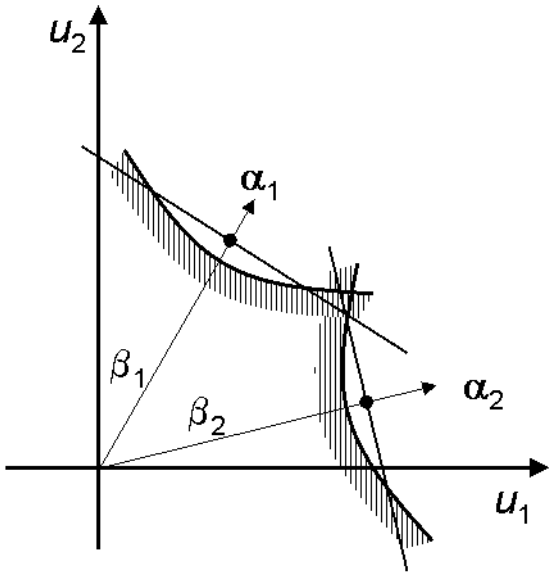


Figure 8.4 Approximation of Large Intersection

The steps in the linearization procedure are:

- *Active constraints*: The single events are first linearized at their separate design points. The linearized version of a single event is as for FORM, however, the distance parameter β is adjusted to account for the second order probability. A single event that has its adjusted linearization point on the event boundary of the resulting linearized intersection is denoted as *active*, see Figure 8.4.

- *Inactive constraints*: The single events which are not active, are denoted as *inactive*. The inactive single events are linearized in succession, if possible. The linearization point of an inactive single event is the point of maximum likelihood on the part of its event boundary that intersects the event defined by the previous linearizations. If this intersection is empty, then the event is deleted from further calculations.

In Figure 8.4 the process is demonstrated for an intersection with two single events. Denoting the linearization of single event no. *i*:

$$g_i(\mathbf{u}) = \beta_i + \mathbf{a}_i^T\mathbf{u} \tag{8.3}$$

the linear surfaces defines a point, $-\mathbf{b}=(-\beta_i)$, on the cumulative multivariate normal distribution, with mean $\mathbf{0}$, standard deviation $\mathbf{1}$ and correlation $\mathbf{R}=(\rho_{ij})=(\mathbf{a}_i^T\mathbf{a}_j)$. Thus

$$P_{\mathrm{SORM}} = 1-\Phi(-\mathbf{b};\mathbf{R}) \tag{8.4}$$

The corresponding reliability index is:

$$\beta_{SORM} = -\Phi^{-1}(P_{SORM}) \tag{8.5}$$

8.3.3 Proban Options

Proban detects whether the intersection is a small intersection or a large intersection. The user optionally chooses ton exclude inactive constraints from the linearization procedure. The pertinent command is DEFINE FORM-SORM.

8.4 Union of Single Events

8.4.1 The Complementary Problem

As for FORM, the union of single events is replaced by the complementary intersection of single events. The SORM for an intersection of single events now applies.

8.4.2 Bounds

Using this method, each single event is replaced by its SORM equivalent linear surface. The bounding technique for FORM is applied to the linearized single events.

8.4.3 Proban Options

In Proban solving the complementary problem is the default method for union of single events. The user optionally chooses a bounds method. The pertinent command is DEFINE FORM-SORM.

8.5 Union of Intersections Single Events

8.5.1 Bounds

The event probability is computed using bounds. Each intersection subsystem is treated by the methods of Section 8.3. The Ditlevsen bounds, see Appendix E, require the probability of the intersection of intersections. This intersection probability is computed from the linearized intersections.

9 NESTED RELIABILITY ANALYSIS

9.1 Method

A nested reliability analysis has the form

$$P_E = P(G((\mathbf{X}_1, P_{SUB}(\mathbf{X}_2)) \leq 0)) \tag{9.1}$$

in which the vectors of random basic variables, $\mathbf{X}_1$ and $\mathbf{X}_2$, may have elements in common.

The probability function P_{SUB} is modelled as a variable, X_p, of type probability. This attribute assigns an event and a probability result to X_p. The probability result is one of P_{SUB}, $\log_{10}(P_{SUB})$ and β_{SUB}-the reliability index corresponding to P_{SUB}.

The variables $\mathbf{X}_2$ on which P_{SUB} depends are all included in the definition of the event. The variables become stochastic in the outer analysis by assigning them as conditioning variables for X_p.

9.1.1 Restrictions

- The top event must be a single event which is not an equality event.
- The method used for calculation of the event probability of X_p is restricted to FORM and SORM.
- The method used for calculation of the outer event is restricted to FORM.
- If the SORM method is used for calculation of P_{SUB}, then a response surface method should be used for the design point search for the outer event. This is necessary since the SORM does not give analytic derivatives of P_{SUB} with respect to P_{SUB}.

10 CROSSING RATE

The rate of a stochastic process, **X**, crossing into an event is calculated as

$$\nu = E\left(\frac{1}{D}\int_{T_S}^{T_S+D} \nu(t, \mathbf{X}_1)dt\right) \tag{10.1}$$

in which $\mathbf{X}=(\mathbf{X}_1,\mathbf{X}_2(t))$, i.e., $\mathbf{X}_1$ is the vector of time independent variables and $\mathbf{X}_2(t)$ is the vector of time dependent variables. The time dependent variables are integrated over to yield $\nu(t, \mathbf{x}_1)$. The expectation E is taken over $\mathbf{X}_1$. T_S is the starting time of the process, and D is its duration.

The crossing rate $\nu(t, \mathbf{x}_1)$ is calculated using FORM or SORM in connection with formulas derived from the parallel system method, Hagen and Tvedt (1990), Ref. /13/.

If the stochastic process is stationary, i.e., the time t is not a variable in the model, then the formula for the crossing rate simplifies to:

$$\nu = E(\nu(\mathbf{X}_1)) \tag{10.2}$$

If a subvector **Y** of $\mathbf{X}_1$ is to be integrated together with $\mathbf{X}_2(t)$, then with $\mathbf{X}_1=(\mathbf{Z},\mathbf{Y})$ there is:

$$\nu = E\left(\frac{1}{D}\int_{T_S}^{T_S+D} \nu(t, \mathbf{Z})dt\right) \tag{10.3}$$

The subvector is assigned to the event by using the command ASSIGN SUBLEVEL INTERGRATION.

11 FIRST PASSAGE PROBABILITY

The probability of a stochastic process, **X**, crossing into an event for the first time in the time interval [T_S, T_S+D] is calculated as

$$P_E = E(P_p(\mathbf{X}_1)) \tag{11.1}$$

in which $\mathbf{X}=(\mathbf{X}_1,\mathbf{X}_2(t))$, i.e., $\mathbf{X}_1$ is the vector of time independent variables and $\mathbf{X}_2(t)$ is the vector of time dependent variables. The expectation E is taken over **X**.

The first passage probability conditioned on the values of $\mathbf{X}_1$, is calculated according to the Poisson assumption

$$P_p(\mathbf{x}_1) = 1-(1-P_0(\mathbf{x}_1))^m \exp\left(-\frac{mn\int_{T_S}^{T_S+D}\nu(t,\mathbf{x}_1)dt}{1-P_0(\mathbf{x}_1)}\right) \tag{11.2}$$

T_S is the starting time of the process, and D is its duration, m yields the extreme value of m independent realizations of the process and n is the number of equal periods of length D. The time dependent variables $\mathbf{X}_2(t)$ are integrated over to yield $\nu(t, \mathbf{x}_1)$.

The crossing rate $\nu(t, \mathbf{x}_1)$ is calculated using FORM or SORM in connection with formulas derived from the parallel system method, Hagen and Tvedt (1990), Ref. /13/.

If the stochastic process is stationary, i.e., the time t is not a variable in the model, then the formula for the first passage probability conditioned on the values of $\mathbf{X}_1$ simplifies to:

$$P_p(\mathbf{x}_1) = 1-(1-P_0(\mathbf{x}_1))^m \exp\left(-\frac{mnD\nu(\mathbf{x}_1)}{1-P_0(\mathbf{x}_1)}\right) \tag{11.3}$$

If a subvector **Y** of $\mathbf{X}_1$ is to be integrated together with $\mathbf{X}_2(t)$, then with $\mathbf{X}_1=(\mathbf{Z},\mathbf{Y})$ the formulas become:

$$P_E = E(P_p(\mathbf{Z})) \tag{11.4}$$

and

$$P_p(\mathbf{z}) = 1-(1-P_0(\mathbf{z}))^m \exp\left(-\frac{mn\int_{T_S}^{T_S+D}\nu(t,\mathbf{z})dt}{1-P_0(\mathbf{z})}\right) \tag{11.5}$$

The subvector is assigned to the event by using the command ASSIGN SUBLEVEL INTERGRATION.

12 APPLICATIONS INVOLVING EQUALITY EVENTS

The single equality event defines a surface in the *x*-space. This surface is defined by the equation

$$G_{SE}(\mathbf{x}) = 0 \tag{12.1}$$

corresponding to the boundary of the event function. If no measured value is assigned to the event, then Proban computes the partial derivative of the top event probability with respect to θ, θ being the threshold value of the equality event. Let E be the top event and $E = \tilde{E} \bigcap \{\mathbf{x}; G_{SE}(\mathbf{x}) = 0\}$ then

$$\frac{\partial P}{\partial \theta} = \frac{\partial}{\partial \theta} P(\tilde{E} \bigcap \{\mathbf{x}; G_{SE}(\mathbf{x}) - \theta \le 0\})_{\theta = 0} \tag{12.2}$$

In case there are k such surfaces

$$G_{SE,\,i}(\mathbf{x}) = 0, \qquad i = 1, 2, \ldots, k \tag{12.3}$$

the formula becomes:

$$\frac{\partial^k P}{\partial \theta_1 \ldots \partial \theta_k} = \frac{\partial^k}{\partial \theta_1 \ldots \partial \theta_k} P(\tilde{E} \bigcap G_{SE,\,1}(\mathbf{x}) - \theta_1 \le 0 \bigcap \cdots \bigcap G_{SE,\,k}(\mathbf{x}) - \theta_k \le 0)_{\theta_1 = \ldots = \theta_k = 0} \tag{12.4}$$

The events of Equation (12.2) and Equation (12.4) are approximated using FORM or SORM. The k surfaces

$$G_{SE,\,i}(\mathbf{X}) - \theta_i = 0, \qquad i = 1, 2, \ldots, k \tag{12.5}$$

are treated as equality constraints during the design point search. The derivative is then obtained through a suitable formula.

13 APPLICATIONS INVOLVING MEASURED VALUES

If a measurement is carried out, then the following limit process defines the meaning of measuring a variable

$$find = \lim_{\Delta \to 0} (a_m - \Delta \le a_m \le a_m + \Delta) \tag{13.1}$$

The process is between a lower bound an upper bound as the distance between the bounds narrows to zero. This defines the process of measuring a variable uniquely.

In Proban the measurement is defined through a single equality event which represents the surface

$$a(\mathbf{X}) = a_m \tag{13.2}$$

and the assignment of the variable representing a_m as the measured value. Notice that the functional form of Equation (13.2) is not required. The method implemented in Proban ensures that the results of the calculations are invariant with reformulations of Equation (13.2), for as long as the surface is preserved.

If there are k measured values, each assigned a single equality event with event function

$$G_{SE,\,i}(\mathbf{x},\, a_{m,\,i}) = 0, \qquad i = 1, 2, \ldots, k \tag{13.3}$$

the formula becomes:

$$\frac{\partial^k P}{\partial\theta_1 \ldots \partial\theta_k} = \frac{\partial^k}{\partial\theta_1 \ldots \partial\theta_k} P(\tilde{E} \bigcap G_{SE,\,1}(\mathbf{x}, a_{m,\,1} + \theta_1) \le 0 \bigcap \cdots \bigcap G_{SE,\,k}(\mathbf{x}, a_{m,\,k} + \theta_k) \le 0)_{\theta_1 = \ldots = \theta_k = 0} \tag{13.4}$$

The events of Equation (13.2) and Equation (13.4) are approximated using FORM or SORM. The k surfaces

$$G_{SE,\,i}(\mathbf{x},\, a_{m,\,i} + \theta_i) = 0, \qquad i = 1, 2, \ldots, k \tag{13.5}$$

are treated as equality constraints during the design point search. The derivative is then obtained through a suitable formula, Hagen et al (1995), Ref. /14/.

14 MEAN VALUE FORM FOR DISTRIBUTIONS

Since Paloheimo and Hannus (1974), Ref. /30/, a number of suggestions to linearize at points other than the design point has been put forward. A suggestion by Wu and Wirsching (1987), Ref. /39/, is to base the choice of linearization point on the gradient of the event function $g(\mathbf{u})$, taken at the origin, $\nabla g(\mathbf{0})$. The function is calculated at points δ_i along the gradient direction. The probability corresponding to a point δ_i is

$$P(g(\mathbf{u}) \le c_i) \approx \Phi(-\delta_i) \tag{14.1}$$

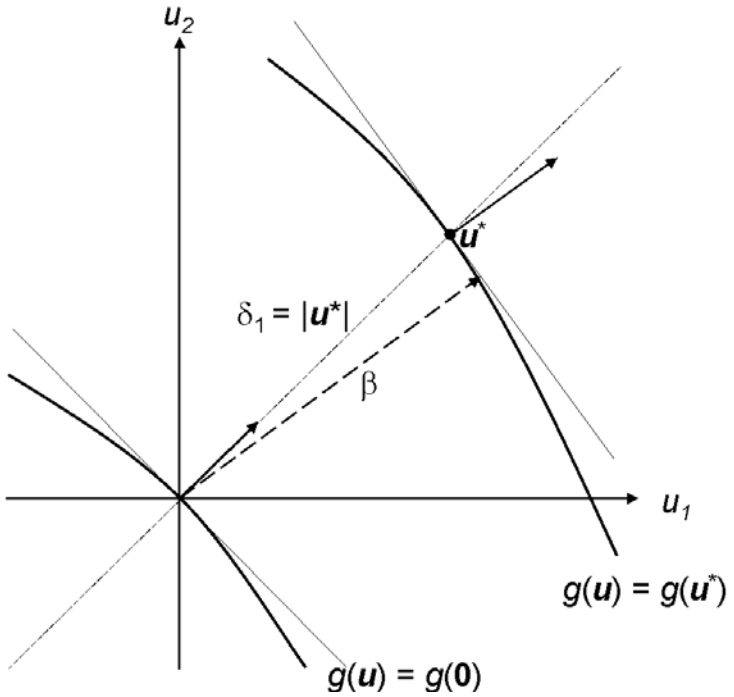


Figure 14.1 Mean-Value FORM for Distributions

If the computation of $g(\mathbf{u})$ is costly, then it may be prohibitively costly to generate the distribution by other methods than the Mean-Value FORM. (This is the only justification for the method.) However, for the results to be useful, the level surfaces, c, of the single event function must be close to linear and nearly parallel in the range of interest. The event function must also be a monotonous function of δ. This is often true for single event functions used in engineering.

A variation of this method, using also the gradients at the end-points of a prescribed probability interval, is also available in Proban.

15 GENERATED DISTRIBUTION

The distribution of the value of a variable for which properties of the distribution are calculated by the use of FORM, is denoted as a generated distribution. Thus, by creating a variable X as generated distribution of the variable Z, the variable X assumes the distribution of Z as calculated by FORM. The variable X can be used like any other variable assigned a distribution from the distributions library.

The variable X can be conditioned on a set of variables, say $\mathbf{Y}$, in the definition of Z. The distribution of $X(\mathbf{y})$ then becomes the distribution of Z with $\mathbf{y}$ as arguments to Z.

The optimization algorithm NLPQL is used with the calculation of the probability $P(X(\mathbf{y})\leq x)$ and the optimization algorithm UNMIN is used with the calculation of the fractile x from $P(X(\mathbf{y})\leq x)=p$. These optimization methods are described in Appendix A.

16 DESIGN POINT SIMULATION

16.1 Motivation

Monte Carlo sampling around the design point is an importance sampling method suggested by Shinozuka (1984), Ref. /34/.

The idea is to find the *u*-space point of maximum likelihood on the event boundary and to perform a Monte-Carlo simulation by sampling *u*-space points around this point. The sampling density is for each variable a normal distribution centred at the design point. The process is demonstrated in Figure 16.1.

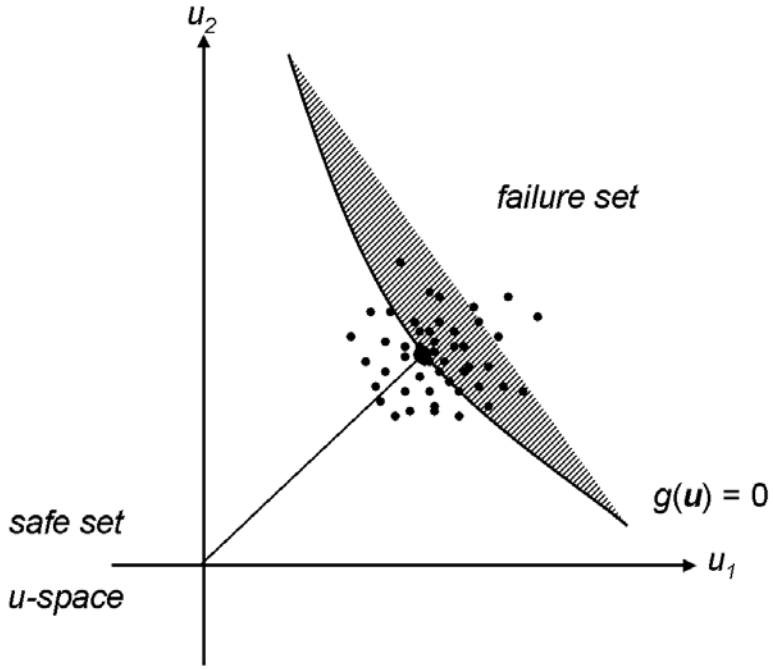


Figure 16.1 Design Point Simulation

The probability is simulated by adding a weighted indicator function for each point.

The Monte-Carlo simulation method as implemented here samples from the joint distribution of n independent normal random variables, $\mathbf{V}$, the sampled point being $\mathbf{v}_i$. The sampled u-space point is $\mathbf{u}_i = \mathbf{d}+\mathbf{v}_i$ where $\mathbf{d}$ is the design point $\mathbf{u}^*$ or optionally a point shifted from the design point. The basic variables are calculated taking the inverse $\mathbf{x}_i=\mathbf{T}^{-1}(\mathbf{u}_i)$, Section 6.2. The indicator function $I(\mathbf{u})$,

$$I(\mathbf{u}) = \begin{cases} 1 & \text{if } g(\mathbf{u}) \le 0 \\ 0 & \text{if } g(\mathbf{u}) > 0 \end{cases} \tag{16.1}$$

is evaluated at each sampled point and the probability is estimated as

$$\hat{P}_E = \frac{1}{N}\sum_{i=1}^{N} I(\mathbf{u}_i)\left(\prod_{j=1}^{n}\sigma_j\right)\frac{\exp\left(-\sum_{j=1} u_{i,j}^2\right)}{\exp\left(-\sum_{j=1}^{n}\left(\frac{u_{i,j}-d_{i,j}}{\sigma_j}\right)^2\right)} \tag{16.2}$$

Here d_i is the design point coordinate no. i and σ_i is the standard deviation of the sampling density. Optionally d_i is a point shifted from the design point.

The advantage of the method is that writing $\mathbf{v} = \mathbf{u}-\mathbf{d}$, then there is

$$\hat{P}_E = \phi(\mathbf{d})\frac{(2\pi)^{n/2}}{N}\sum_{i=1}^{N} I(\mathbf{u}_i)\left(\prod_{j=1}^{n}\sigma_j\right)\exp\left(-\mathbf{d}^T\mathbf{v}_i - \sum_{j=1}^{n} v_{i,j}^2\left(1-\frac{1}{\sigma_j^2}\right)\right) \tag{16.3}$$

so that one effectively simulates a multiplicative factor to the term $\phi(\mathbf{d})$. If each standard deviation is equal to 1 and $\mathbf{d}$ is the design point, then

$$\hat{P}_E = \phi(\mathbf{u}^*)\frac{(2\pi)^{n/2}}{N}\sum_{i=1}^{N} I(\mathbf{u}_i)\exp(-(\mathbf{u}^*)^T\mathbf{v}_i) \tag{16.4}$$

17 MONTE-CARLO SIMULATION

17.1 Simulation of Probability

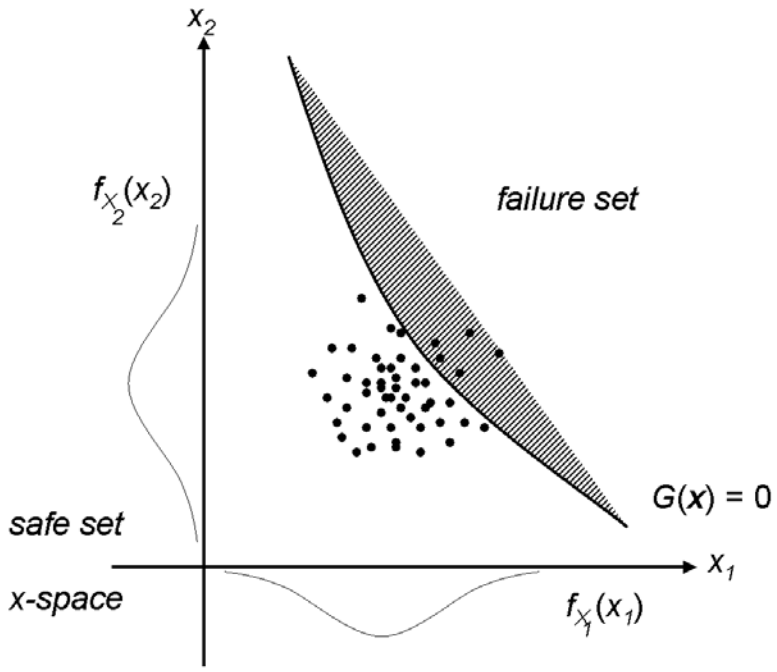


Figure 17.1 The Monte-Carlo Simulation Method

The Monte-Carlo simulation method as implemented in Proban samples from the joint distribution of the n random basic variables as shown in Figure 17.1. This is done by sampling from n independent standard normal distributions and then taking the inverse $\mathbf{x}_i=\mathbf{T}^{-1}(\mathbf{u})$, Section 6.2. Here $\mathbf{u}_i$ is the sampled **u**-space point. The indicator function $I(\mathbf{x})$,

$$I(\mathbf{x}) = \begin{cases} 1 & \text{if } G(\mathbf{x}) \le 0 \\ 0 & \text{if } G(\mathbf{x}) > 0 \end{cases} \tag{17.1}$$

is evaluated at each sampled point and the probability is estimated as

$$\hat{P}_E = \frac{1}{N}\sum_{i=1}^{N} I(\mathbf{x}_i) \tag{17.2}$$

An advantage of the method is that it makes use of point values of the event function only. Thus, the event function is not required to be a smooth function of its variables. Also, the estimate of the event probability is unbiased.

17.2 Adjusted Simulation of Probability

There are implemented two ways of adjusting the sampling density. The first method samples a point in u-space as follows:

$$u_j = \mu_j + \sigma_j y_j \tag{17.3}$$

where y_j is sampled as a standard normal distributed variable. The probability is estimated as the weighted sum:

$$\hat{P}_E = \frac{1}{N}\sum_{i=1}^{N} I(\mathbf{x}_i)\left(\prod_{j=1}^{n}\sigma_j\right)\frac{\exp\left(-\sum_{j=1} u_{i,j}^2\right)}{\exp\left(-\sum_{j=1}^{n}\left(\frac{u_{i,j}-\mu_{i,j}}{\sigma_j}\right)^2\right)} \tag{17.4}$$

The second method introduces strata for the sampled variable. The strata are entered as probability points for the sampled variable. Two consecutive points define an interval that may optionally be weighted. The weights are scaled so that their sum is one. The distribution of weights is sampled from as described in the following. First a probability p_u is sampled from the uniform distribution. Then the scaled weights are summed until the sum exceeds p_u. Let the lower and upper probability points for this interval of the ith sample be $p_{i,l}$ and $p_{i,u}$. Let the summed scaled weight at the lower probability point be $w_{i,l}$ and the scaled weight of the interval be Δw_i. Then the corresponding point p_s on the stratified distribution is:

$$p_s = \frac{(p_u - w_{i,l})}{\Delta w_i}(p_{i,u} - p_{i,l}) + p_{i,l} \tag{17.5}$$

The fractile x_i of the distribution corresponding to p_s is then calculated and the probability is estimated as

$$\hat{P}_E = \frac{1}{N}\sum_{i=1}^{N} I(\mathbf{x}_i)\frac{\Delta w_i}{(p_{i,u} - p_{i,l})} \tag{17.6}$$

This scales the probability so that a the region outside the probability intervals contributes zero to the probability. In Proban the probability intervals are entered as an increasing sequence of optionally weighted probabilities. By letting the sequence start with p>0, then the interval from zero to p contributes zero to the

probability. By letting the sequence end with $p<1$, then the interval from p to 1 contributes zero to the probability.

Similarly one may enter the intervals as an increasing sequence of fractiles. The region from $-\infty$ to the first fractile contributes zero to the probability, and so does the region from the last region to ∞. In Proban a probability interval corresponding to a fractile interval is calculated by using the fractiles at **u=0** to calculate the distribution parameters that are themselves random variables. These probability intervals are used in the simulation as above.

17.3 Simulation of Distribution

The distribution of the value of a variable is generated by ordering the sample

$$y_i = y(\mathbf{x}_i), \qquad i = 1, 2, \ldots, N \tag{17.7}$$

where y is the value of the variable, $\mathbf{x}_i$ is the ith sample of the random variables and N is the number of samples.

17.4 Options for Simulation of Probability

Maximum number of points to be sampled.

Coefficient of variation for the probability.

Maximum CPU-time.

17.5 Options for Simulation of Distribution

Maximum number of points to be sampled.

18 DIRECTIONAL SIMULATION

18.1 Motivation

The idea of directional simulation was put forward by Deak (1981), Ref. /6/, who used it to sample a multi-normal probability. The method samples directions uniformly distributed on the surface of the *n*-dimensional unit hyper-sphere centred at the origin of the *u*-space, Figure 18.1. The method assembles the contributions to the probability integral conditioned on the sampled directions. A number of variance reducing methods, i.e. sampling of orthogonal sets of directions, were proposed.

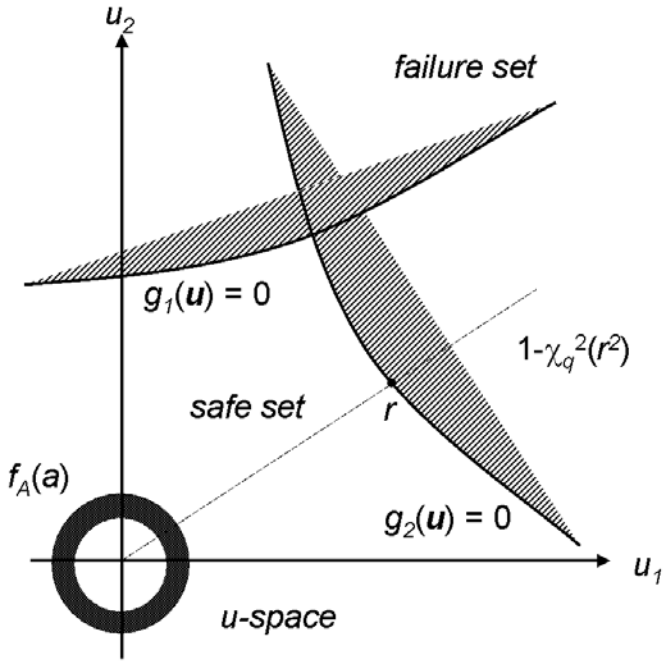


Figure 18.1 Directional Simulation

Application of the method to general systems with non-linear event boundaries was suggested by Bjerager (1988), Ref. /2/. The implementation of the method in Proban follows this paper.

The motivation for the method is that it allows unbiased and efficient sampling of small probabilities, provided that n is not too large.

18.2 Sampling Formula

The probability of the event may be formulated as

$$P_E = \int_{g(u\mathbf{a}) \le 0} f_{\chi_n^2}(\nu) d\nu d\Omega \tag{18.1}$$

in which ν is a χ_n^2-distributed random variable, $\mathbf{a}$ is a unit vector and $d\Omega$ is the surface element of the n-dimensional unit sphere. The unit directions $\mathbf{a}_i$ are sampled and the function χ_n^2 conditioned on $\mathbf{a}$ is integrated. This is done by finding the upper bound $\nu_{u,i}$ and the lower bound $\nu_{l,i}$ of the intervals where $g(u\mathbf{a}) \le 0$ and by adding these contributions to the integral.

$$P(\mathbf{a}_i) = \sum_{i=1}^{m} (\chi_n^2(\nu_{u,i}) - \chi_n^2(\nu_{l,i})) \tag{18.2}$$

assuming there are m such intervals. The estimator for the probability is

$$\hat{P}_E = \frac{1}{N} \sum_{i=1}^{N} P(\mathbf{a}_i) \tag{18.3}$$

where there is N sampled directions.

The simplest method that reduces the variance of the simulated values is to replace $P(\mathbf{a})$ by

$$P_1(\mathbf{a}) = \frac{1}{2}(P(\mathbf{a}) + P(-\mathbf{a})) \tag{18.4}$$

A further reduction of the variance of the sampled set is obtained by sampling of a randomly oriented orthonormal system of vectors $\mathbf{b}_1$, $\mathbf{b}_2$,..., $\mathbf{b}_n$. A new system of vectors is formed as follows: To each set $\{\mathbf{b}_{i_1}, \mathbf{b}_{i_2}, \ldots, \mathbf{b}_{i_n}\}$ of k out of n vectors, there are $m = \frac{n!}{k!(n-k)!}$ sets, the following 2^k vectors $\mathbf{a}_i(\mathbf{b})$, are formed

$$\mathbf{a}_i(\mathbf{b}) = \frac{1}{\sqrt{k}}(s_1\mathbf{b}_{i_1} + s_2\mathbf{b}_{i_2} + \ldots + s_n\mathbf{b}_{i_n}) \tag{18.5}$$

with $s_j = \pm 1$, j=1,2,...,k. One point is sampled as

$$O_k = \frac{1}{2^k m} \sum_{i=1}^{m} \sum_{\{\mathbf{s}; s_j = \pm 1, j = 1, 2, \ldots, k\}} P(\mathbf{a}_i(\mathbf{b})) \tag{18.6}$$

The estimator for the probability based on sampling N orthonormal systems of vectors, is

$$\hat{P}_{E,k} = \frac{1}{N} \sum_{i=1}^{N} O_{k,j} \tag{18.7}$$

Taking advantage of symmetry, the number of directions calculated, r, is half the number indicated above. Thus

$$r = \frac{n!}{k!(n-k)!}2^{k-1} \tag{18.8}$$

The number of function calls is thus approximately rn_a, where n_a is the average number of function calls required to perform the summation of Equation (18.2).

18.3 Options

The following options for choice of estimator are implemented:

DEFAULT The selection of estimator is based on the u-space dimensions as follows

Use $\hat{P}_{E,1}$ if $n \leq 2$ or $16 \leq n \leq 50$

Use $\hat{P}_{E,2}$ if $n=3$ or $11 \leq n \leq 15$

Use $\ddot{P}_{E,3}$ if $4 \leq n \leq 10$

RANDOM-DIRECTION The method uses $P_I(a)$ of Equation (18.4).

ORTHOGONAL-1 The method uses $\hat{P}_{1,k}$ of Equation (18.7).

ORTHOGONAL-2 The method uses $\hat{P}_{2,k}$ of Equation (18.7).

ORTHOGONAL-3 The method uses $\hat{P}_{3,k}$ of Equation (18.7).

The following options for search of the intervals of Equation (18.2) are available:

RISKY-AND-FAST The sign of $g(\mathbf{u})$ at $\mathbf{u}=\mathbf{0}$ is compared to the sign of $g(u\mathbf{a})$ at a point, $u_{\max}$, determined by the demand of accuracy in the solution. If the signs are opposite, it is assumed that $g(u\mathbf{a})$ has exactly one zero point in the direction of $\mathbf{a}$ in the interval $[0,u_{\max}]$. If the signs are the same, then $g(u\mathbf{a})$ has no zero points in this interval.

MEDIUM-SAFE Calculates the function $g(u\mathbf{a})$ at a sequence of equidistant user defined steps for increasing u until a sign shift of the function is detected or the maximum, $u_{\max}$, is reached. If a sign shift is detected, then the point $g(u\mathbf{a})=0$ is found. It is assumed that this is the only zero point for the direction.

SAFE-AND-SLOW Steps out with equidistant user defined steps and checks for sign shifts of the function $g(u\mathbf{a})$. All contributory intervals will be picked up, unless the step-length is so large that two solutions are bypassed in one step.

STEP Step length in u-space used in the search for zero points.

SEARCH LIMIT The maximum u-space distance searched in the direction of $\mathbf{a}$. The user may define this as the corresponding probability of the excluded interval or directly as the max-

imum *u*-space distance searched. When a zero point is found, the maximum *u*-space distance searched is further reduced to $P(u>u_{max}) = P(u>u_{zero\ point})*1.0\text{-}7$

19 AXIS-ORTHOGONAL SIMULATION

19.1 Motivation

Axis-orthogonal sampling around the design point is an importance sampling method suggested by Hohenbichler and Rackwitz (1984), Ref. /22/, and Schall et al. (1986), Ref. /33/.

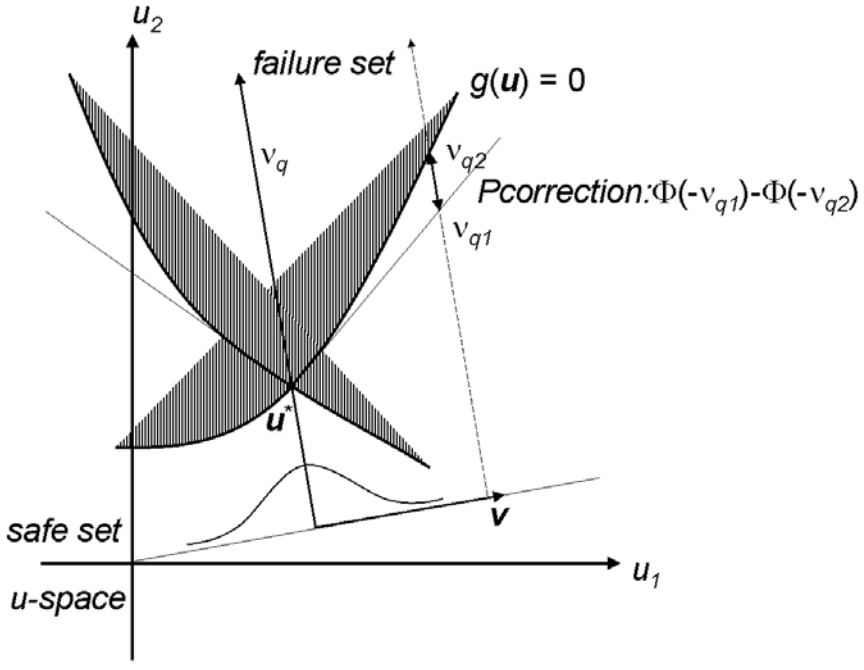


Figure 19.1 Axis-Orthogonal Simulation

The idea is to define an axis for a small intersection domain, see Section 7.3.1, and to define a sampling density in a plane orthogonal to this axis. The method assumes that the true boundary of the event is suitably

approximated by a set of linear surfaces obtained by using the FORM linearization of small intersection domains. The probability of the linearized domain is obtained by methods available for the multinormal distribution and to correct this probability with a simulated estimate of the probability difference of the true event and the linearized event.

Since the cumulative multinormal distribution is computed only approximately for dimensions greater than one, using the methods of Appendix D, the axis-orthogonal simulation method estimates an approximation to the true probability. The quality of this approximation is governed by the quality with which the multinormal integral is calculated.

In addition to the simulation of the additive correction above, there is implemented also a multiplicative correction, Bjerager (1988), Ref. /2/.

19.2 Selection of Axis

Let $\mathbf{a}_i$, i=1,2,...,n be the set of normalized gradients for the single events active at the design point of the intersection. The axis, $\mathbf{a}_a$, is the averaged gradient, pointing into the interior of at the design point, Figure 19.1:

$$\mathbf{a}_a = -\frac{\sum_{i=1}^{n} \mathbf{a}_i}{\left| \sum_{i=1}^{n} \mathbf{a}_i \right|} \tag{19.1}$$

The coordinate system is rotated so that the new coordinate u_n is in the direction of the axis, while $\tilde{\mathbf{u}} = (u_1, u_2, ..., u_{n-1})$ are the coordinates normal to u_n. The axis is now defined as

$$\mathbf{a}_u = \tilde{\mathbf{u}}_0 + u_n \mathbf{e}_n \tag{19.2}$$

where $\tilde{\mathbf{u}}_0$ is the design point in the new coordinates and $\mathbf{e}_n$ is the unit vector in direction of u_n.

19.3 Sampling of a Multiplicative Correction

The sampling density, H_M, is as follows: a) In the space spanned by the gradients of the m single events active at the design point, the m-dimensional standard normal density, conditioned on the linearized event, is used, Bjerager (1988), Ref. /2/. b) In the directions orthogonal to this space, and normal to the axis, $\mathbf{a}_u$, $n-m-1$ independent standard normal random variables centred on the axis are sampled from. Thus

$$P_E = P_L \int_{g(\mathbf{u}) \le 0} \frac{P_A(\tilde{\mathbf{u}})}{\Phi(-u_n(\tilde{\mathbf{u}}))} H_M(\tilde{\mathbf{u}}) d\tilde{\mathbf{u}} = P_L C \tag{19.3}$$

in which P_L is the probability of the linearized domain, C is the multiplicative correction, $u_n(\tilde{\mathbf{u}})$ is the intersection of the linearized domain and the line

$$\mathbf{l}(u_n) = \tilde{\mathbf{u}} + u_n \mathbf{e}_n \tag{19.4}$$

$P_A(\tilde{\mathbf{u}})$ is the true probability of the event conditioned on the line defined by $\mathbf{l}(u_n)$. The estimator for C is

$$\hat{C} = \frac{1}{N}\sum_{i=1}^{N}\frac{P_A(\tilde{\mathbf{u}}_i)}{\Phi(-u_n(\tilde{\mathbf{u}}_i))} \tag{19.5}$$

19.4 Sampling of an Additive Correction

The sampling density is in the n-1 dimensional standard normal density function $\phi(\tilde{\mathbf{v}})$ with $\tilde{\mathbf{u}} = \tilde{\mathbf{u}}_0 + \tilde{\mathbf{v}}$, i.e. with centre at the design point and normal to the axis $\mathbf{a}_a$. The linearized domain is bounded by the linear approximations of the single events active at the design point and the linear approximations of the constraints having a positive $\mathbf{a}_i^T\mathbf{a}_a$. The difference of the probability of the true event and the probability of its linear approximation is estimated. Thus

$$P_E = P_L \int_{g(\mathbf{u})\le 0} (P_A(\tilde{\mathbf{u}}) - \Phi(-u_n(\tilde{\mathbf{u}})))\frac{\phi(\tilde{\mathbf{u}})}{\phi(\tilde{\mathbf{v}})}\phi(\tilde{\mathbf{v}})d\tilde{\mathbf{v}} = P_L + A \tag{19.6}$$

The estimator for A is

$$\hat{A} = \frac{1}{N}\sum_{i=1}^{N}(P_A(\tilde{\mathbf{u}}_i) - \Phi(-u_n(\tilde{\mathbf{u}}_i)))\frac{\phi(\tilde{\mathbf{u}}_i)}{\phi(\tilde{\mathbf{v}}_i)} \tag{19.7}$$

19.5 Computation of P_A

The integral estimator are implemented:

$$P_A(\tilde{\mathbf{u}}) = \int_{g(\tilde{\mathbf{u}},u_n)\le 0} \phi(u_n)du_n \tag{19.8}$$

The integral is evaluated through a search for the points $u_n(l_i)$ and $u_n(u_i)$ denoting the lower and upper bounds of the intervals where $g(\tilde{\mathbf{u}}, u_n)$ is less than zero, and then to sum up the contributions using the formula

$$P_A(\tilde{\mathbf{u}}) = \sum_{i=1}^{m}(\Phi(u_n(u_i)) - \Phi(u_n(l_i))) \tag{19.9}$$

assuming there are m such intervals. The following options for search of the intervals of Equation (19.9) are available:

RISKY-AND-FAST — The sign of $g(\tilde{\mathbf{u}}, u_n)$ at $\mathbf{u}=\mathbf{0}$ is compared to the sign at $u_n=u_{n,max}$ determined by the demand of accuracy in the solution. If the signs are opposite, it is assumed that $g(\tilde{\mathbf{u}}, u_n)$ has exactly one zero point in the direction of u_n in the interval $[0,u_{n,max}]$. If the signs are the same, then $g(\tilde{\mathbf{u}}, u_n)$ has no zero points in this interval.

MEDIUM-SAFE — Calculates the function $g(\tilde{\mathbf{u}}, u_n)$ at a sequence of equidistant user defined steps for increasing u_n until a sign shift of the function is detected or the maximum, $u_{n,max}$, is reached. If a sign shift is detected, then the point $g(\tilde{\mathbf{u}}, u_n)=0$ is found. It is assumed that this is the only zero point for the direction.

RISKY-AND-FAST — Steps out with equidistant user defined steps and checks for sign shifts of the function $g(\tilde{\mathbf{u}}, u_n)$. All contributory intervals will be picked up, unless the step-length is so large that two solutions are bypassed in one step.

20 LATIN-HYPERCUBE SIMULATION

20.1 Method

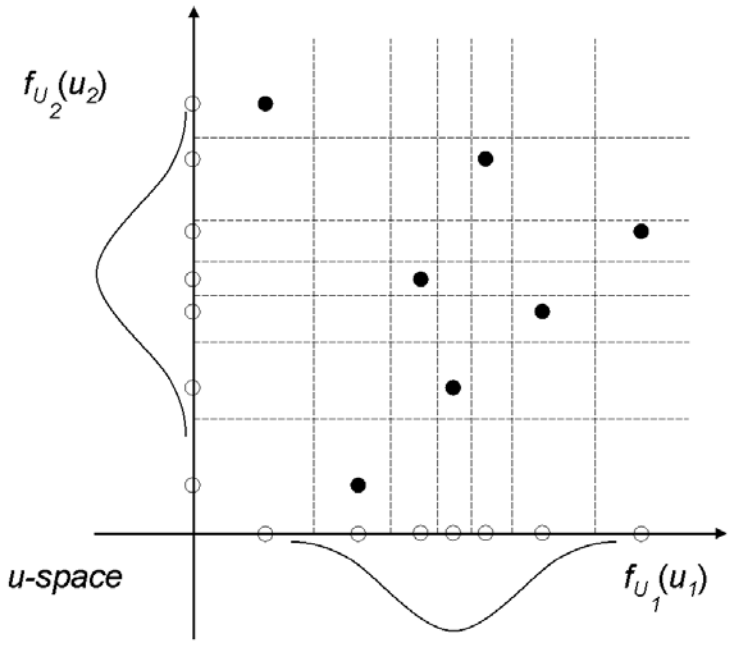


Figure 20.1 The Latin-Hypercube Simulation Method

The Latin-hypercube simulation method divides each u-space coordinate axis u_i into m segments, $I_i(j)$, j=1,2,...,m, of equal probability, p=1/m. One point is sampled from each of the segments according to the standard normal distribution. The ith axis is thus represented by the points u_{ij}, j=1,2,...,m. The first point in the Latin-Hypercube sample is generated by sampling one value $u_{ij,1}$from each axis i. The second point, $u_{ij,2}$, of the sample is generated the same way, except that the segments containing $u_{ij,1}$ are deleted from the

sample. And so on. This way the m points of one Latin-Hyperube are sampled, Mc Kay et al. (1979), Ref. / 28/. An example with n=2 and m=7 is shown in Figure 20.1. The expected number of Latin-Hypercube points to hit an event is m/P_E, the same as for the Monte Carlo method, however, the variance is in general less.

The Latin-hypercube method is used to sample the central part of the distribution of a variable.

20.2 Options for Simulation of Probability

The number of hypercubes to be sampled.

The number of points per hypercube.

21 PARAMETRIC SENSITIVITY FOR PROBABILITY AND CROSSING RATE

21.1 Introduction

The parametric sensitivity is here discussed for the probability of an event without a time dependent stochastic process in the model. The result for the first passage probability, the crossing rate and the derivative of a probability is, however, similar.

The definition of the parametric sensitivity factor is

$$P_\theta = \frac{\partial P_E}{\partial \theta} = \lim_{\Delta\theta \to 0} \frac{P_E(\theta + \Delta\theta) - P_E}{\Delta\theta} \tag{21.1}$$

where P_E is the probability of the event E.

Parametric sensitivity factors can be used to estimate the change in probability resulting from an increment $\Delta\theta$ of θ. Thus

$$P_E(\theta + \Delta\theta) \approx P_E + \Delta\theta P_\theta \tag{21.2}$$

The sensitivity factors for the probability are given also for the reliability index, β_E, and the logarithm of the probability, $\log_{10}(P_E)$. The sensitivity factor for the reliability index is generally more predictive than the sensitivity factor for the probability.

Two methods are available for the calculation of the sensitivity factor, the asymptotic method and the analytic method. The method differs in the way intersections of single events are treated. Everything else is treated the same way.

The asymptotic method was developed by Hohenbichler (1984), Ref. /20/, and Hohenbichler and Rackwitz (1986), Ref. /21/. The formulas implemented in Proban are outlined below.

The analytic method yields analytic derivatives for the FORM probability of intersections of single events. The method was developed by Bjerager and Krenk (1987), Ref. /1/, Bjerager and Krenk (1989), Ref. /3/ and Madsen (1988), Ref. /27/.

The analytic method yields analytic correspondence between P_E and the partial derivatives P_θ which yields analytic derivatives in a nested reliability analysis when FORM is used for the inner loop analysis.

21.2 Single Event

The *u*-space formulation of the event function is $g(\mathbf{u},\theta)$. The FORM sensitivity factor for the reliability index is

$$\beta_{\theta,\,\mathrm{FORM}} = \frac{\frac{\partial}{\partial\theta}g(\mathbf{u}^*,\,\theta)}{|\nabla g(\mathbf{u}^*,\,\theta)|}, \qquad \beta_{\mathrm{FORM}} \to \infty \tag{21.3}$$

The expression $\frac{\partial}{\partial\theta}g(\mathbf{u}^*,\,\theta)$ is computed using numerical differentiation.

When the SORM method is used, then the parametric sensitivity becomes:

$$\beta_{\theta,\,\mathrm{SORM}} = \frac{\phi(\beta_{\mathrm{SORM}})}{\phi(\beta_{\mathrm{SORM}})}\frac{\Phi(-\beta_{\mathrm{SORM}})}{\Phi(-\beta_{\mathrm{SORM}})}\beta_{\theta,\,\mathrm{FORM}}, \qquad \beta_{\mathrm{FORM}} \to \infty \tag{21.4}$$

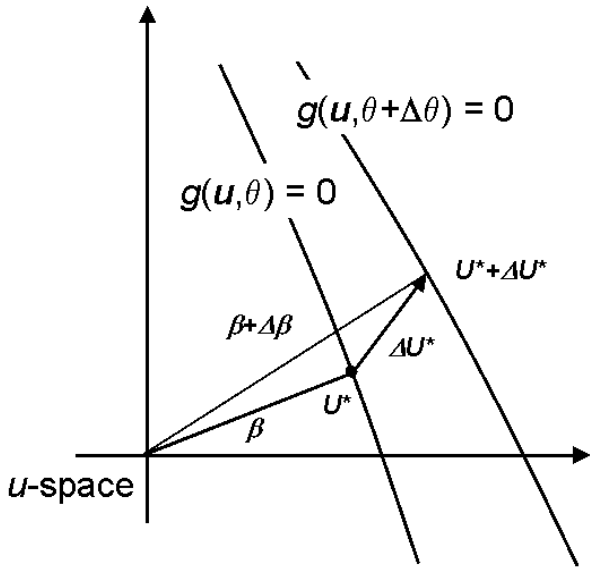


Figure 21.1 Design Point for θ and $\theta + \Delta\theta$

21.3 Small Intersection

The asymptotic formula treats the linearized event as a multinormal integral where the correlation matrix is kept fixed during differentiation. The integration limits may, however, depend on the parameter. The parametric sensitivity for the reliability index is

$$\beta_\theta = \frac{\Phi(-\beta_E)}{\phi(\beta_E)} \sum_{i \in A} \frac{\frac{\partial}{\partial \beta_i}\Phi(-\mathbf{b};\mathbf{R})}{\Phi(-\mathbf{b};\mathbf{R})} \frac{\partial \beta_i}{\partial \theta}, \qquad |\mathbf{u}^*| \to \infty \tag{21.5}$$

The elements of **b** are the reliability indices of the single event linearizations at **u*** and β_E is either of β_{FORM} and β_{SORM}.

The method of Equation (21.5) neglects the influence of inactive constraints, the derivatives of the coefficient of correlation. The analytic result accounts for this, but require the additional calculation of $\mathbf{G}(\mathbf{u}^*,\theta)$ and $\frac{\partial}{\partial \theta}\nabla g(\mathbf{u}, \theta)$.

21.4 Large Intersection

The asymptotic formula treats the linearized event as the complementary probability of a multinormal integral where the correlation matrix is kept fixed during differentiation. The integration limits may, however, depend on the parameter. The parametric sensitivity for the reliability index is

$$\beta_\theta = \frac{\Phi(-\beta_E)}{\phi(\beta_E)} \sum_{i \in A} \frac{-\frac{\partial}{\partial \beta_i}\Phi(-\mathbf{b};\mathbf{R})}{1-\Phi(-\mathbf{b};\mathbf{R})} \frac{\partial \beta_i}{\partial \theta}, \qquad |u_i| \to \infty, i \in A \tag{21.6}$$

with definitions as for formula Equation (21.5).

21.5 Union

The union of *m* single events and the union of *m* intersections of single events are treated the same way:

$$\beta_\theta \approx k \sum_{i=1}^{m} \phi(\beta_i)\beta_{\theta,i} \tag{21.7}$$

The asymptotic result assumes that the norm, $|\mathbf{u}^*|$, of each subsystem design point tends to ∞.

Using

$$\Phi(-\beta_S) = \sum_{i=1}^{m} \Phi(-\beta_i) \tag{21.8}$$

then *k* is:

$$k = \frac{\Phi(-\beta_E)}{\Phi(-\beta_S)}\phi(\beta_S) \tag{21.9}$$

where β_E is the reliability index which corresponds to the upper bound of the probability.

If there is a union of single events, then by solving the complementary problem, the formulation for large intersection is used, as described above.

22 PARAMETRIC SENSITIVITY FOR DISTRIBUTION

The parametric sensitivity factor is for the mean, μ, the standard deviation, σ, the skewness, δ, and the kurtosis, κ, of the distribution:

The definition of the parametric sensitivity factor for the mean is

$$\frac{\partial \mu}{\partial \theta} = \lim_{\Delta\theta \to 0} \frac{\mu(\theta + \Delta\theta) - \mu}{\Delta\theta} \tag{22.1}$$

and similar for the three other statistical quantities.

The distribution of a variable, the statistical moments of the distribution ant the parametric sensitivity factors are computed simultaneously. The computation of the sensitivity factors require $\frac{\partial}{\partial \theta} g(\mathbf{u}, \theta)$ to be computed at each sampled point.

APPENDIX A OPTIMIZATION

A 1 NLPQL

NLPQL is a Non-Linear Programming method which solves a sequence of Quadratic, Linearly constrained sub problems. The problem solved is:

$$\begin{aligned} &\textit{minimize} \;\; f(\mathbf{u}) \\ &\textit{subject to} \;\; g_i(\mathbf{u}) = 0 \qquad i = 1, 2, \ldots, m_e \\ &\qquad\qquad g_i(\mathbf{u}) \le 0 \qquad i = m_e + 1, m_e + 2, \ldots, m \\ &\qquad\qquad u_{L,j} \le u_j \le u_{U,j} \; j = 1, 2, \ldots, n \end{aligned} \tag{A.1}$$

Here $f(\mathbf{u})$ is the objective function, in Proban essentially equal to $|\mathbf{u}|$, however modified so that the derivatives are continuous at $\mathbf{u}=\mathbf{0}$. The number of (usually non-linear) equality constraints is m. The vector $\mathbf{u}_L$ stores the lower bounds and the vector $\mathbf{u}_U$ stores the upper bounds of the n variables $\mathbf{u}$.

Each iteration comprises an estimation of a search direction (requiring one evaluation of the constraint gradients) and one line iteration to find the minimum of the search function.

The Kuhn-Tucker optimality criterion is

$$\begin{aligned} \nabla f(\mathbf{u}^*) + \sum_{i \in A} \lambda_i \nabla g_i(\mathbf{u}^*) &= \mathbf{0} \\ \sum_{i \in A} \lambda_i g_i(\mathbf{u}^*) &= 0 \end{aligned} \tag{A.2}$$

in which A is the set of constraints active at the optimality point and $\lambda_i, i \in A$ is the corresponding set of Lagrange parameters. (The method implemented excludes inactive constraints, i.e., those inequality constraints that have $g(\mathbf{u}^*)>0$. Also, the contribution from variables reaching a bound is omitted from the presentation.) The quantities checked for are

- Sum of constraint violations, *SCV*, of the active constraints:

$$SCV = \sum_{i \in A} |g_i(\mathbf{u}^*)| \tag{A.3}$$

- Norm of the gradient of the Lagrangian, *NLG*:

$$NLG = \max\left(\left| \frac{\partial}{\partial u_j} f(\mathbf{u}) + \sum_{i \in A} \lambda_j \frac{\partial}{\partial u_j} g_i(\mathbf{u}^*) \right|, j = 1, 2, \ldots, n \right) \tag{A.4}$$

- Kuhn-Tucker optimality criterion, *KTO*:

$$KTO = \sum_{i \in A} |\lambda_i g_i(\mathbf{u}^*)| \tag{A.5}$$

The convergence criterion, ε, is checked as follows:

$$\begin{aligned} KTO &\le \varepsilon \\ SCV &\le \varepsilon^{1/2} \\ NLG &\le \varepsilon^{1/4} \end{aligned} \tag{A.6}$$

The search for an optimal point is performed on an augmented Lagrangian function. The default search method is BFGS, which is a quasi Newton method that builds a sequence of quadratic approximations to the Lagrangian from gradients. Optionally, the steepest descent method may be selected.

The search method determines a direction of improvement for the Lagrangian function. The step length in this direction is by default unrestricted (option free). Optionally the user may specify an upper limit for the step length in order to avoid overshooting (option value).

If constraints are poorly scaled, then the *KTO* and the *NLG* criteria may be fulfilled, while the constraint violation criterion, *SCV*, is not satisfied. Therefore this criterion may optionally be excluded from the convergency test or be given a user defined value.

In some analyses, for example in nested reliability analysis, it is important that the analysis continues to run even if the inner loop optimization fails to converge. Optionally one may therefore specify that the optimizer delivers the best point it found rather than stopping the calculations. When a design point search fails it is often true that the search converges, but the stop criterion is too strict. The best point option may be useful also in this case. However, the results of such an analysis should be carefully examined, since there are situations in which the best point option yields an erroneous result.

In addition to this, one can change the maximum number of general iterations on the augmented Lagrangian and the maximum number of function evaluations during line search for determination of the step length.

A 2 The SQP optimizer

The Sequential Quadratic Programming optimizer, SQP, is an old version of the NLPQL method. The convergence criteria are as for NLPQL, but the options are limited to ε, Appendix A.6, the maximum number of general iterations and the maximum number of function evaluations during line search.

A 3 The RSM optimizer

The Response Surface Method optimizer works for single events only. The method approximates the *u*-space event function by using information from the last *m* iterations, $\mathbf{u}_i$. Then the *u*-space design point is found for the approximation surface. Convergency is checked for. If not converged, then new information about the event function is obtained around the current design point, a new approximation surface is built and the process is repeated.

The values calculated around design point $\mathbf{u}_i$ are, if two-way incrementation is used:

$$h_j^i = \begin{cases} g(\mathbf{u}_i), & j = 0 \\ g(\mathbf{u}_i + \Delta_i \mathbf{e}_k), & j = k,\ k = 1, \ldots, n \\ g(\mathbf{u}_i - \Delta_i \mathbf{e}_k), & j = n + k,\ k = 1, \ldots, n \end{cases} \tag{A.7}$$

If one-way incrementation is used, then only the n+1 first values are calculated. The new function is built from the points where

$$\left|\mathbf{u}_i - \mathbf{u}_k\right| \le r_i \tag{A.8}$$

where r_i is the radius (range) of contribution, i.e., all the function values generated around design points closer to $\mathbf{u}_i$ than r_i contribute to the approximation. The current design point is $\mathbf{u}_i$.

If a reduction factor different from 1 is employed, then Δ_i is divided once per iteration from its initial value by the reduction factor until its minimum value is reached.

The function built takes the form:

$$\tilde{g}(\mathbf{u}) = b + \mathbf{c}^T\mathbf{u} + \frac{1}{2}\mathbf{u}^T\mathbf{G}\mathbf{u} \tag{A.9}$$

If a linear surface is generated, then **G** is assumed zero. If a linear + diagonal of quadratic is generated, then the off-diagonal elements of **G** are assumed zero. If a linear + quadratic function is generated, then the off diagonal elements are also estimated.

If the unit direction vector at the initial iteration point is **a**, then the squares of the lowest values are summed up until the zero constant limit is reached, but not passed, the index set being *I*. The *u*-space coordinates in the sum are set to zero for the rest of the analysis and the final reliability index is adjusted with the omission factor as follows:

$$\beta_{\text{FORM}} = \beta_{\text{FORM,Surface}}\left(1 - \sum_{i \in I} a_i^2\right)^{-1/2} \tag{A.10}$$

This is useful if an analysis is costly and it is necessary to reduce the problem size as much as possible, for example if a FEM code is employed.

A 4 The RFCRC optimizer

This optimizer is restricted to a single event. It employs a Racwitz-Fiessler method, (the projected gradient method suggested for use with structural reliability calculations). If the next step is less than the previous step divided by a reduction factor, then the projected gradient method is used for the next step. If not, a circle step is employed. The projected gradient step is:

$$\mathbf{s}_i = -\frac{\nabla g(\mathbf{u}_i)}{(\nabla g(\mathbf{u}_i))^2} g(\mathbf{u}_i) \tag{A.11}$$

The next iterate is:

$$\mathbf{u}_{i+1} = \mathbf{u}_i + \mathbf{s}_i \tag{A.12}$$

The reduction factor is

$$r_i = \frac{|\mathbf{s}_{i-1}|}{|\mathbf{s}_i|} \tag{A.13}$$

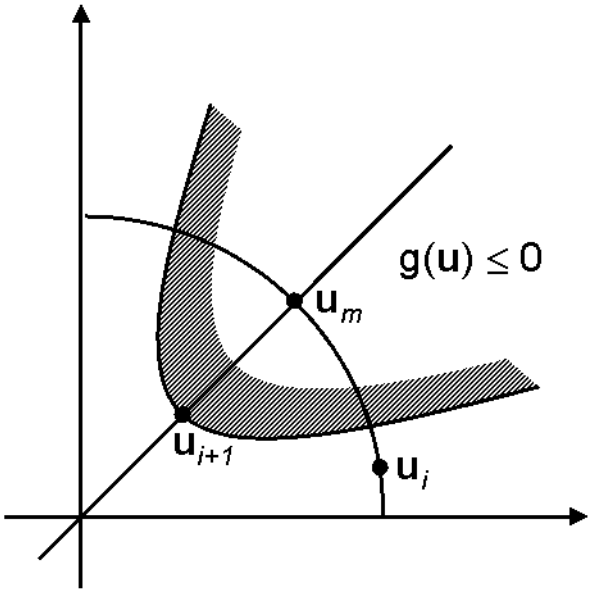


Figure A.1 Circle Step

If this factor is greater than a user defined limit, then the method converges too slowly, or not at all. Then the step is replaced by a circle step, Figure A.1 which finds the minimum value point, $\mathbf{u}_m$, for the event function along a circle with radius $\mathbf{u}_i$, and finds the zero point for the event function along the line $t\mathbf{u}_m$. This procedure may converge also when there is a notch at the design point.

A 5 The UNMIN optimizer

The UN-constrained MINimization algorithm is implemented for generated distributions only. It minimises the value of a function on a sphere. It is used to calculate a fractile of a generated distribution from the probability.

The options are maximum number of iterations, maximum number of function evaluations during line search and the stop criterion.

APPENDIX B CODING OF A FUNCTION

B 1 General

The coding of a function should be carried out with due consideration of the optimization algorithm to be used. The following is important:

- Differentiability of the function
- Computation cost
- Consistent model

B 2 Differentiability

The NLPQL optimizer uses function values and gradients to build a quadratic approximation to an (augmented) Lagrange function. Each function should therefore be twice differentiable in the region of interest for the optimization.

The RFCRC optimizer uses function values and gradients to define a direction plane for the next step. The function should therefore differentiable in the region of interest for the optimizer. The method tolerates a notch at the optimal point if only the direction plane is well approximated during the iterations.

The RSM optimizer is based on function values only. Since it fits a linear or quadratic surface to the *u*-space function, the function should be well approximated by such surfaces.

If the gradient is poorly estimated, the NLPQL optimizer may not converge. The RFCRC optimizer will converge, but slowly.

Some pitfalls and their remedies are given in the following.

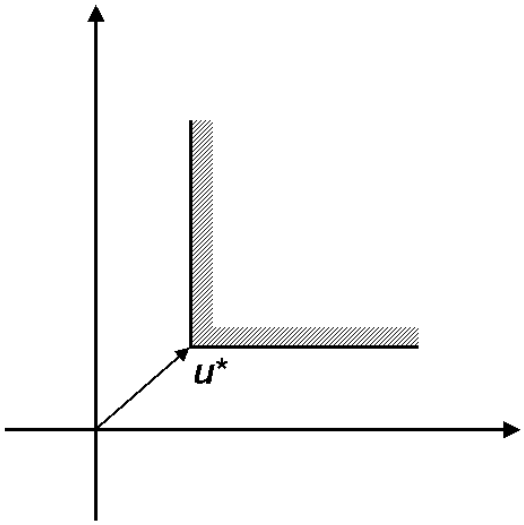


Figure B.1 Non-differentiable at the Design Point

- Function *non-differentiable at design point*: A typical error is to model a system of event functions as one single event function, for example, the intersection event in Figure B.1. This event function is not differentiable at the design point. The NLPQL optimizer therefore fails. The RFCRC optimizer will find the design point, but the FORM or SORM will give poor results for the reliability estimate. The RFCRC optimizer must therefore be used in conjunction with a simulation method, for example the design point simulation method.

  Such problems should be reformulated into a system of event functions. The intersection of Figure B.1 can then be solved using NLPQL and good approximations to the probability can be calculated using FORM or SORM.

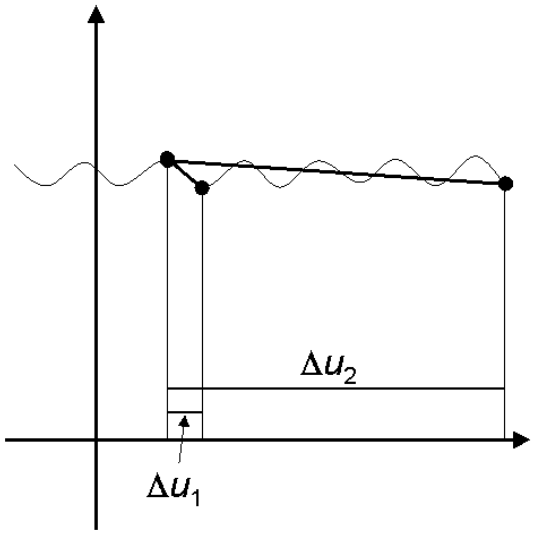


Figure B.2 Rugged Function

- *Rugged function*: This situation is shown in Figure B.2. A small differentiation increment Δu_1 yields a derivative dominated by numeric noise. By using a larger differentiation increment, Δu_2, the numeric noise is of less significance. However, if the function is strongly non-linear, increase in differentiation increments may yield poor gradients. The best way to overcome this problem, is to code the analytic derivative of the function as well, and use the option ANALYTIC for the calculation of derivatives.

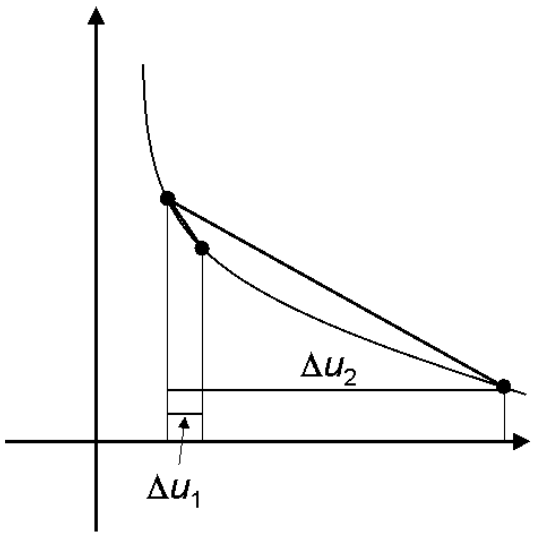


Figure B.3 Strongly Non-linear Function

- *Strongly non-linear function*: This situation is shown in Figure B.3. A large differentiation increment Δu_2 yields a poor derivative dominated by curvature. By using a smaller differentiation increment, Δu_1, the

curvature may be of less significance. However, the derivative may then be dominated by numeric noise. In this situation one may use the option TWOWAY-INCREMENTATION to estimate the derivative. This yields an error in the derivative of $O((\Delta u)^2)$ and therefore allows larger increments than the option ONE-WAY-INCREMENTATION, which yields an error in the derivative of order $O(\Delta u)$.

A better method is to smooth the function, if possible. If the event function can be written as

$$G(\mathbf{x}) = r(\mathbf{x}_1) - l(\mathbf{x}_2) \tag{B.1}$$

with $\mathbf{x}=(\mathbf{x}_1,\mathbf{x}_2)$, then the following methods are available:

- If $r>0$ and $l>0$ always, then

  $$\tilde{G}(\mathbf{x}) = \log(r(\mathbf{x}_1)) - \log(l(\mathbf{x}_2)) \tag{B.2}$$

  preserves the zero surface for the function. Therefore the probability estimates and sensitivity factors are the same as for $G(\mathbf{x})$.

- If r arbitrary and/or l arbitrary always, then

  $$\tilde{G}(\mathbf{x}) = \operatorname{sgn}(r(\mathbf{x}_1))\left|r(\mathbf{x}_1)\right|^{\xi} - \operatorname{sgn}(l(\mathbf{x}_2))\left|l(\mathbf{x}_2)\right|^{\xi} \tag{B.3}$$

  preserves the zero surface for the function. Therefore the probability estimates and sensitivity factors are the same as for $G(\mathbf{x})$. The parameter ξ can be modelled as a fixed valued variable, allowing the user to experiment to find a suitable value of ξ.

- If $r>0$ and $l>0$ always, then

  $$\operatorname{logt}(y) = \begin{cases} -1 - \log(-y) & y < -1 \\ y & -1 \le y \le 1 \\ 1 + \log(y) & 1 < y \end{cases} \tag{B.4}$$

  $$\tilde{G}(\mathbf{x}) = \operatorname{logt}(r(\mathbf{x}_1)) - \operatorname{logt}(l(\mathbf{x}_2))$$

  preserves the zero surface for the function. Therefore the probability estimates and sensitivity factors are the same as for $G(\mathbf{x})$.

An often occurring error is to use sloppy tolerances on the results computed by sub-programs called by the function routine. The results from such programs should have a high precision, for example 6 significant digits. (The Proban routines that calculates the Rosenblatt transformation and its inverse produce 10 or more significant digits.) An often occurring error is the use of sloppy tolerances on the precision in the results from iterative procedures, for example numerical integration.

B 3 Consistent Model

A consistent model has the property that the joint probability distribution of the random basic variables has positive values only in the x-domain where the event function is defined. Thus, a consistent model is defined for all points in u-space.

If the model is not defined consistently, there is a positive probability for an undefined part of the model. If a sequence of iterates hits a point where the event function is not defined, (ISTAT=1 is returned from a function subroutine), the optimizer stops. The remedy is to make the model consistent.

APPENDIX C QUADRATIC FORMS IN NORMAL SPACE

C 1 Parabola

The distribution of the parabolic form

$$M_P = -Y_n + \sum_{j=1}^{n-1} \lambda_j Y_j^2 \tag{C.1}$$

is considered in Tvedt (1988), Ref. /35/, and in Tvedt (1989), Ref. /36/. In Tvedt (1989), Ref. /36/, it is shown how to obtain the parabolic form M_P from the parabolic approximation to the u-space g-function in the vicinity of the design point. The Laplace transform of M_P is

$$G_{M_P} = \exp\left(\frac{1}{2}u^2\right) \prod_{j=1}^{n-1} (1 - 2\lambda_j u)^{-1/2} \tag{C.2}$$

Using $\beta=\beta_{FORM}$, the probability of the parabolic domain is

$$P_P = P(M_P \leq -\beta) \tag{C.3}$$

The density integral and the probability integral of M_P are computed using the saddle point integration method to evaluate the inversion integrals. Note that this procedure yields the true result for both the density integral for M_P and the probability integral for M_P.

C 2 Second Order Taylor Expansion

The distribution of the quadratic form

$$M_T = \nu + \sum_{j=1}^{n-1} (\gamma_j Y_j + \lambda_j Y_j^2) \tag{C.4}$$

is considered in Tvedt (1989), Ref. /36/. In Tvedt (1989), Ref. /36/, it is shown how to obtain the quadratic form M_T from the second order Taylor expansion of the *u*-space *g*-function in the vicinity of the design point. The Laplace transform of M_T is

$$G_{M_T} = \exp(\nu u) \prod_{j=1}^{n-1} \frac{\exp\left(\frac{1}{2}\frac{\gamma_j^2 u^2}{1-2\lambda_j u}\right)}{(1-2\lambda_j u)^{1/2}} \tag{C.5}$$

Using $\beta=\beta_{FORM}$, the probability of the quadratic domain is

$$P_T = P(M_T \le -\beta) \tag{C.6}$$

The density integral and the probability integral of M_T are computed using the saddle point integration method to evaluate the inversion integrals, Rice (1980), Ref. /31/, and Helstrom (1983), Ref. /17/. Note that this procedure yields the true result for both the density integral for M_T and the probability integral for M_T.

C 3 Asymptotic Results

Two asymptotic results, i.e. as $\beta\to\infty$, Breitung (1984), Ref. /4/, and Hohenbichler (unpublished), are implemented. Breitung's result reads

$$P_P = \Phi(-\beta) \prod_{i=1}^{n-1} (1 + 2\lambda_i \beta)^{-1/2} \tag{C.7}$$

Hohenbichler's result reads

$$P_P = \Phi(-\beta) \prod_{i=1}^{n-1} \left(1 + 2\lambda_i \frac{\varphi(\beta)}{\Phi(-\beta)}\right)^{-1/2} \tag{C.8}$$

APPENDIX D MULTINORMAL INTEGRAL

Whenever the probability of an intersection of single events is calculated using approximate methods, the multinormal integral is involved, i.e. the cumulative probability of the multinormal integral must be calculated. Denoting the n integration limits as $\mathbf{x}$, and the correlation matrix as $\mathbf{C}$, the integral is

$$\Phi(\mathbf{x};\mathbf{C}) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\cdots\int_{-\infty}^{x_n} \frac{1}{(2\pi)^{n/2}|\mathbf{C}|^{1/2}} \exp\left(-\frac{1}{2}\mathbf{s}^T\mathbf{C}^{-1}\mathbf{s}\right) ds_1 ds_2 \ldots ds_n \tag{D.1}$$

where |C| is the determinant of C. For dimensions $n \geq 4$, there is no tractable method available to compute this integral with arbitrary precision, except for spacial cases, Madsen et. al. (1986), Ref. /24/, Chapter 4.5.1. However, an approximation method that gives good results also for higher dimensions is available, Hohenbichler (1982), Ref. /19/, and Gollwitzer and Rackwitz (1988), Ref. /12/. The multinormal probability can be expressed as the probability of the intersection of the n single linear events

$$\left\{\mathbf{u}; g_i(\mathbf{u}) = \sum_{j=1}^{i} L_{ij}u_j \leq x_i\right\}; i = 1, 2, \ldots, n \tag{D.2}$$

The matrix L is the lower triangular Cholesky factor of $\mathbf{C}$, i.e. $\mathbf{L}\mathbf{L}^{\mathrm{T}}=\mathbf{C}$ and $\mathbf{U}$ is a set of n independent standard normal random variables. The method is based on the following reduction formula, Hohenbichler (1983), Ref. /19/:

$$P\left(\bigcap_{i=1}^{n} g_i(\mathbf{u}) \leq 0\right) = \Phi(x_1) P\left(\bigcap_{i=2}^{n} g_i(\tilde{\mathbf{u}}) \leq 0\right) \tag{D.3}$$

where $\tilde{u}_i = u_i; i = 2, 3, \ldots, n$ and

$$u_1 = \Phi^{-1}(\Phi(x_1)\Phi(\tilde{u}_1)) \tag{D.4}$$

The idea is to approximate the probability of the reduced parallel system by a new multinormal integral of dimension $n-1$, $\Phi(\mathbf{x}^1;\mathbf{C}^1)$, and a correction factor K:

$$P\left(\bigcap_{i=2}^{n} g_i(\tilde{\mathbf{u}}) \le 0\right) \approx K\Phi(\mathbf{x}^1;\mathbf{C}^1) \tag{D.5}$$

using the SORM method for intersections, Gollwitzer and Rackwitz (1988), Ref. /12/. The elements of $\mathbf{x}^1$ are the new integration limits and the elements of $\mathbf{C}^1$ are the new correlations. Then the integral $\Phi(\mathbf{x}^1;\mathbf{C}^1)$ is reduced the same way. The procedure is repeated until $n=1$.

APPENDIX E BOUNDS ON PROBABILITY OF A UNION

E 1 Union

Consider a union of n events, E_i, i=1,2,...,m, each occurring with probability P_i. The union of the events is

$$E = \bigcup_{i=1}^{m} E_i \tag{E.1}$$

and the probability is

$$P_E = P\left(\bigcup_{i=1}^{m} E_i\right) \tag{E.2}$$

The probability P_E is calculated from the probabilities P_i and the intersection probabilities $P_{ij}=P(E_i \cap E_j)$ in terms of bounds. A first order bound makes use of P_i only. A second order bound makes use also of P_{ij}.

E 2 Simple Bounds

The *simple lower bound* on P_E, $P_{E,L}$ is the maximum probability of a single event

$$P_{E,L} = \max\{P_i, i = 1, 2, \ldots, m\} \tag{E.3}$$

The *simple upper bound* on P_E, $P_{E,U}$ is the sum of the single event probabilities:

$$P_{E,U} = \min\left\{\sum_{i=1}^{m} P_i, 1\right\} \tag{E.4}$$

E 3 Ditlevsen Bounds

The Ditlevsen bounds, Ditlevsen (1979), Ref. /10/, Kounias (1968), Ref. /23/, are second order bounds, the *Ditlevsen lower bound* on P_E, $P_{E,L}$ is:

$$P_{E,L} = P_1 + \sum_{i=2}^{m} \max\left\{0, P_i - \sum_{j=1}^{i-1} P_{ij}\right\} \tag{E.5}$$

The *Ditlevsen upper bound* on P_E, $P_{E,U}$ is:

$$P_{E,U} = P_1 + \sum_{i=2}^{m} (P_i - \max\{P_{ij}, j = 1, 2, \ldots, i-1\}) \tag{E.6}$$

E 4 Reliability Index for Bounds

The *lower bound on the reliability index* is:

$$\beta_{E,L} = -\Phi^{-1}(P_{E,U}) \tag{E.7}$$

The *lower bound on the reliability index* is:

$$\beta_{E,U} = -\Phi^{-1}(P_{E,L}) \tag{E.8}$$

APPENDIX F TERMINOLOGY

F 1 Introduction

The terminology used in Proban is given in the following sections. Appendix F 2 gives the terminology for events. Appendix F 3 gives the terminology for variables. Appendix F 4 gives the terminology for measures of probability. Appendix F 5 gives the terminology for sensitivity and importance. Appendix F 6 gives the terminology for event approximations. Appendix F 7 gives the terminology for simulation methods. Appendix F 8 gives the definition of statistical quantities used in this manual.

F 2 Events

Event function

The event function, $G(\mathbf{x})$, defines the *event* $G<0$, the *event boundary* $G=0$, and the *complementary event* $G>0$. The *event type* associated with a given event function is often specified, e.g. *single event* function.

Single event, SE

The event is assigned a random variable, a threshold and a relation of the variable to the threshold. The event function is

$$G(\mathbf{x}) = z(\mathbf{x}) - \theta$$

The corresponding event is

$$E_{SE}(\mathbf{x}) = \{\mathbf{x}; G(\mathbf{x}) \le 0\}$$

Intersection of Single Events, ISE

The event function value is the maximum of m single event function values,

$$G_{ISE}(\mathbf{x}) = \max\{G_{SE,i}(\mathbf{x}); i = 1, 2, \ldots, m\}$$

The corresponding event is the intersection of single events

$$E_{ISE}(\mathbf{x}) = \{\mathbf{x}; G_{ISE}(\mathbf{x}) \le 0\} = \bigcap_{i=1}^{m} E_{SE,i}(\mathbf{x})$$

Union of Single Events, USE

The event function value is the minimum of m single event function values,

$$G_{USE}(\mathbf{x}) = \min\{G_{SE,i}(\mathbf{x}); i = 1, 2, \ldots, m\}$$

The corresponding event is the union of single events

$$E_{USE}(\mathbf{x}) = \{\mathbf{x}; G_{USE}(\mathbf{x}) \le 0\} = \bigcup_{i=1}^{m} E_{SE,i}(\mathbf{x})$$

Union of Intersections of Single Events, UISE

The event function value is the minimum of m intersection event function values,

$$G_{UISE}(\mathbf{x}) = \min\{G_{ISE,i}(\mathbf{x}); i = 1, 2, \ldots, m\}$$

The corresponding event is the union of intersections of single events

$$E_{UISE}(\mathbf{x}) = \{\mathbf{x}; G_{UISE}(\mathbf{x}) \le 0\} = \bigcup_{i=1}^{m} E_{ISE,i}(\mathbf{x})$$

Union of Events, UE

The event function value is the minimum of m subevent function values,

$$G_{UE}(\mathbf{x}) = \min\{G_{E,i}(\mathbf{x}); i = 1, 2, \ldots, m\}$$

The corresponding event is the union of subevents

$$E_{UE}(\mathbf{x}) = \{\mathbf{x}; G_{UE}(\mathbf{x}) \le 0\} = \bigcup_{i=1}^{m} E_{E,i}(\mathbf{x})$$

Intersection of Events, IE

The event function value is the maximum of m subevent function values,

$$G_{IE}(\mathbf{x}) = \max\{G_{E,i}(\mathbf{x}); i = 1, 2, \ldots, m\}$$

The corresponding event is the intersection of subevents

$$E_{IE}(\mathbf{x}) = \{\mathbf{x}; G_{IE}(\mathbf{x}) \le 0\} = \bigcap_{i=1}^{m} E_{E,i}(\mathbf{x})$$

F 3 Variables

Random variable	A variable which is uncertain.
Basic variable	A variable in the model space.
Random basic variable	A basic variable which is also a random variable.
Fixed variable	A basic variable which is non-random.
Normalized random variable	The standard normal random variable, *u*-space variable, which the random basic variable is mapped onto, e.g. using the Rosenblatt transformation.
x-space	The space of basic variables.
u-space	The space of independent standard normal variables the basic random variables are mapped onto.
v-space	The space of dependent standard normal variables the basic random variables are mapped onto by use of the Nataf model.

F 4 Probability Measures

Event probability

The probability that the event occur, i.e.

$$P_E = \int_{G(\mathbf{x}) \le 0} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$

where $G(\mathbf{x})$ is the event function and $f_{\mathbf{X}}(\mathbf{x})$ is the joint distribution of the random basic variables.

Reliability

The probability of the complementary event, i.e. $Q_E=1-P_E$.

Reliability index

The argument of the standard normal distribution corresponding to the reliability, i.e.

$$\beta_R = \Phi^{-1}(Q_E) = -\Phi^{-1}(P_E)$$

The reliability index is often indexed with the method used to estimate β_R, for example β_{FORM} and β_{SORM}.

F 5 Parametric Sensitivity and Uncertainty Importance

Parametric sensitivity - probability

The derivative of the probability with respect to a parameter:

$$P_\theta = \frac{dP_E}{d\theta}$$

and similar for the reliability index.

Parametric sensitivity - moments

The derivative of the four first moments of the distribution of a random variable with respect to a parameter, i.e.

$$\gamma_\theta = \frac{d\gamma}{d\theta}$$

where γ is one of μ, σ, δ and κ, i.e., the mean, standard deviation, skewness and kurtosis.

Uncertainty importance

The derivative of the reliability index with respect to the mean of a *u*-space variable:

$$a_\mu = \frac{d\beta_R}{d\mu}$$

In Proban print the uncertainty importance is scaled to give an impression of the percentage of the reliability index which stems from the variable.

$$\tilde{a}_\mu = 100 a_\mu^2$$

F 6 Approximation of Event

Linearization point

The point at which an event function is approximated by a linear/quadratic surface.

Design point

The point on the event boundary which has the highest probability density in *u*-space. This is also the *u*-space event boundary point with the minimum distance to the *u*-space origin.

Active single events

Single events that are active at the design point.

Inactive single events

Single events that are not active at the design point.

FORM

First Order Reliability Method. The event boundaries of active single events are linearized at the design point. Each inactive single event is linearized (in succession) at the point of maximum likelihood on its event boundary, restricted to the event defined by the previously linearized events.

SORM

Second Order Reliability Method. As FORM, however, the curvature of the event boundary in the vicinity of a design point is accounted for through a quadratic form. (Parabolic form for

intersections. Parabolic, diagonal or second order Taylor for single events.)

F 7 Simulation Methods

Monte Carlo	Samples points according to the distribution of the basic random variables. Probability is calculated from evaluation of an indicator function at each sampled point. The density distribution is generated from saving the simulated values and ordering them in an ascending values.
Design point	Finds the design point. Then conducts a Monte Carlo sampling in *u*-space around the *u*-space design point. Probability is calculated from evaluation of a scaled indicator function at each sampled point.
Axis-orthogonal	Finds the design point. Defines an axis for a single event or a small intersection event. Then conducts a Monte Carlo sampling of points in the space orthogonal to the axis. Calculates a correction to the FORM probability by adding up weighted correction probabilities for lines parallel to the axis through the sampled points.
Directional	Samples directions in *u*-space uniformly distributed on the surface of the unit hyper-sphere centred at the u-space origin. A weighted sum of the event probabilities conditioned on the directions sampled is accumulated.
Latin-hypercube	A stratified sampling scheme where each variable is sampled *m* times and these outcomes are randomly combined to form *m* samples.

F 8 Statistical Quantities

Table explains some basic statistical quantities.

Density	$f(x)$	The *density distribution* of the random variable *X*.
Cumulative	$F(x)=\int_{-\infty}^{x} f(s)ds$	The *probability distribution* of the random variable *X*.
Moment	$E((X-a)^n)=\int_{-\infty}^{\infty}(s-a)^n f(s)ds$	The *n*th moment of *X* about *a*.

Mean	$\mu = E(X)$	The *expectation* or *average* of X.
Median	m	The value x so that $F(x) = 0.5$
Standard Deviation	$\sigma = E((X-\mu)^2)^{0.5}$	Measures the *spread* of the distribution of X about μ.
Coefficient of Variation	$cov = \frac{\sigma}{\lvert\mu\rvert}, \mu \neq 0$	
Skewness	$\delta = \frac{E((X-\mu)^3)}{\sigma^3}$	Measures the *symmetry* of the distribution of X about μ. The Normal distribution has skewness equal to 0.
Kurtosis	$\kappa = \frac{E((X-\mu)^4)}{\sigma^4}$	Measures the *peakedness* of the distribution of X about μ. The Normal distribution has kurtosis equal to 3.

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