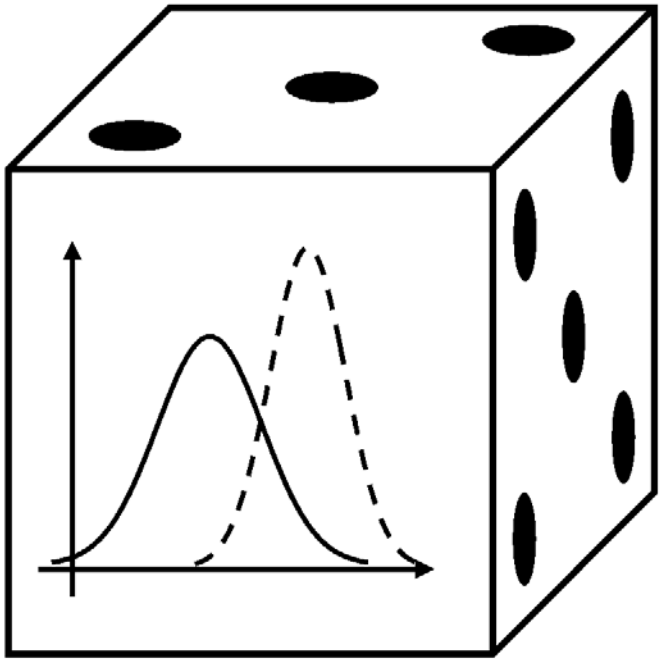
SESAM

USER MANUAL

Proban Distributions



Continuous, Discrete and Correlated Distributions

DET NORSKE VERITAS

SESAM
User Manual

Proban Distributions

Continuous, Discrete and Correlated Distributions

Octber 1st, 2004

Valid from program version 4.4

Developed and marketed by
DET NORSKE VERITAS

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1 INTRODUCTION

The Proban Distributions Manual describes the distribution models available for Proban variables.

- Twenty eight standard distributions are available in the Proban Distributions Library.
- A function of random variables may be treated as a univariate distribution type. This distribution type is then generated by Proban.
- The results of a distribution analysis may be fitted to a number of the distribution types available in the Proban Distributions Library.
- A random variable may be modelled to take the distribution of the maximum or the minimum of an integer number of independent realisations of the basic random variable.
- Correlation of univariate random variables is modelled assuming a Nataf distribution model.
- A continuous time dependent stochastic process is defined in terms of a random variable and its time derivative.
- An argument of a random variable may itself be modelled as a random variable. This permits distribution models in which parameters like mean and standard deviation are random variables. Therefore quite complex conditional distribution models can be defined.

The command CREATE VARIABLE defines the input of random variables and their distribution types. A variable is identified by name.

A variable is a vector if the corresponding distribution is multivariate. A coordinate is referred to by name-index.

The command ASSIGN EXTREME-VALUE allows for the maximum or minimum extreme value input.

The command ASSIGN-CORRELATION defines the correlation input.

The command ASSIGN CONTINUOUS-PROCESS defines the input of a time dependent stochastic process.

The commands CREATE-VARIBLE, ASSIGN EXTREME-VALUE, ASSIGN CORRELATION, and ASSIGN COCONTINUOUS-PROCESS are described in the Proban User Manual.

A typical CREATE VARIABLE situation is shown in Figure 1.1.

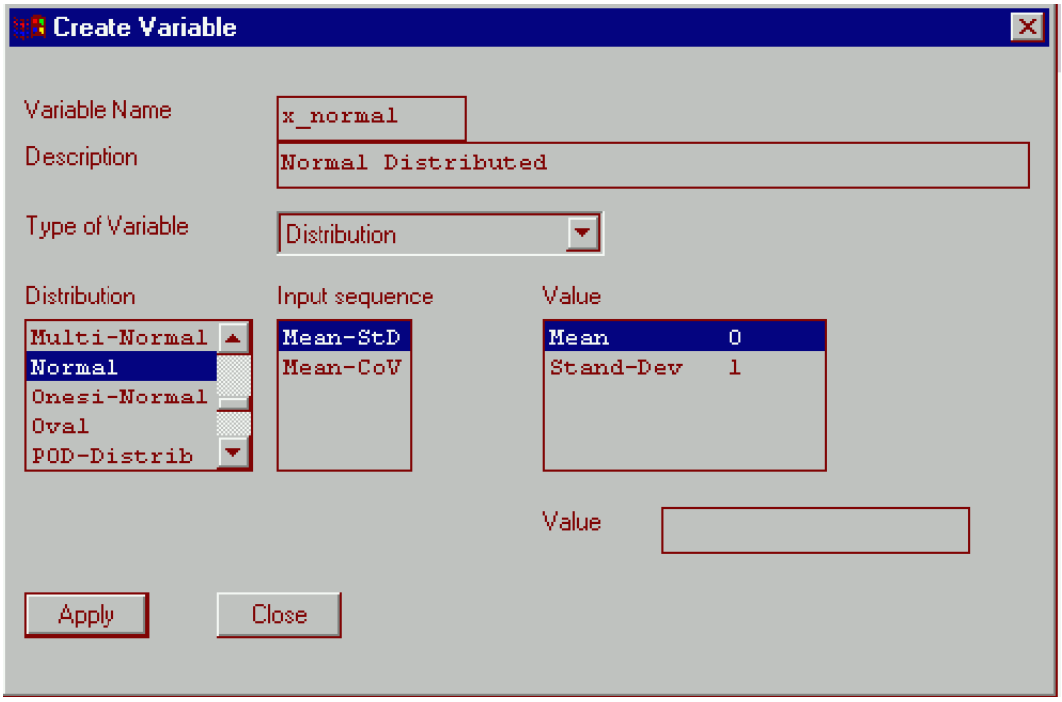


Figure 1.1 Create a Random Variable

1.1 Terminology

Table explains some basic statistical quantities.

Density	$f(x)$	The *density distribution* of the random variable X.
Cumulative	$F(x)=\int_{-\infty}^{x} f(s)ds$	The *probability distribution* of the random variable X.
Moment	$E((X-a)^n)=\int_{-\infty}^{\infty}(s-a)^n f(s)ds$	The nth moment of X about a.
Mean	$\mu = E(X)$	The *expectation* or *average* of X.

Median	m	The value x so that $F(x) = 0.5$
Standard Deviation	$\sigma = E((X-\mu)^2)^{0.5}$	Measures the *spread* of the distribution of X about μ.
Coefficient of Variation	$cov = \frac{\sigma}{\lvert\mu\rvert}, \mu \neq 0$	
Skewness	$\delta = \frac{E((X-\mu)^3)}{\sigma^3}$	Measures the *symmetry* of the distribution of X about μ. The Normal distribution has skewness equal to 0.
Kurtosis	$\kappa = \frac{E((X-\mu)^4)}{\sigma^4}$	Measures the *peakedness* of the distribution of X about μ. The Normal distribution has kurtosis equal to 3.

Table is about standard functions used in the text.

Beta	B(r,s)	$\int_0^1 t^{r-1}(1-t)^{s-1}dt$
B-spline	$B_{i,k,t}(x)$	B-spline no. i of n of order k defined on $t = t_j; j = 1, 2, \ldots, n+k; t_j < t_{j+1}$
Complementary Incomplete Gamma Function	$\Gamma(a,x)$	$\int_x^\infty s^{a-1}e^{-s}ds$
Gamma Function	$\Gamma(a)$	$\int_0^\infty s^{a-1}e^{-s}ds$
Incomplete Gamma Function	$\gamma(a,x)$	$\int_0^x s^{a-1}e^{-s}ds$
Inverse Complementary Incomplete Gamma Function	$\Gamma^{-1}(a,x)$	Solves y from $x = \Gamma(a,y)$.
Inverse Incomplete Gamma Function	$\gamma^{-1}(a,x)$	Solves y from $x = \gamma(a,y)$.
Inverse Standard Normal Distribution Function	$\Phi^{-1}(x)$	Solves y from $x = \Phi(a,y)$.
Riemann Zeta function	$\zeta^{-1}(x)$	$\sum_{n=1}^{\infty} n^{-x}$

Standard Normal Density function	$\varphi(x)$	$\frac{1}{\sqrt{2\pi}} exp\left(-\frac{x^2}{2}\right)$
Standard Normal Distribution Function	$\Phi(x)$	$\int_{-\infty}^{x} \varphi(s) ds$
Euler Constant	γ_e	0.577215664...

1.2 Proban in the SESAM System

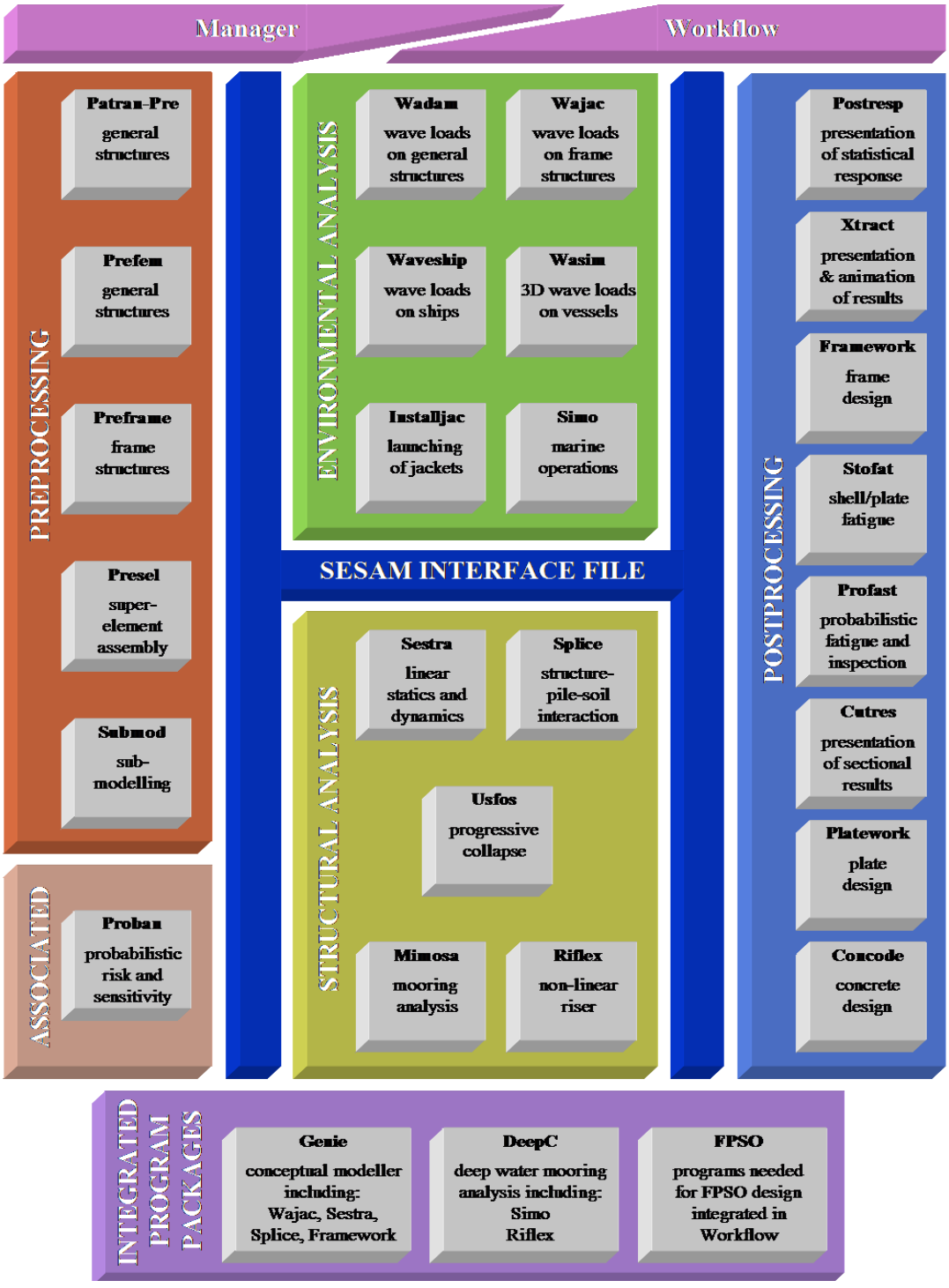


Figure 1.2 SESAM Overview

1.3 How to Read this Manual

Chapter 2 DISTRIBUTION LIBRARY- CONTINUOUS DISTRIBUTIoNS explains each continuous distribution in the Proban Distribution Library.

Chapter 3 DISTRIBUTION LIBRARY - DISCRETE DISTRIBUTIONS explains each discrete distribution in the Proban Distribution Library.

Chapter 4 OTHER DISTRIBUTIONS explains the Generated Distribution, The Max and Min Extreme Value Distributions, Correlation, Conditional Distributions and the Time-dependent Stochastic process.

1.4 Changes from the Previous Revision

The Burr distribution has been added to the Proban Distribution Library.

2 DISTRIBUTION LIBRARY- CONTINUOUS DISTRIBUTIONS

2.1 Introduction

The Proban Distribution Library is a built-in collection of distributions. Each distribution has a *name*, e.g., the Normal distribution has the name *Normal*. A distribution may have more than one *parameter set* (*input sequence*), e.g., the Normal distribution has the parameter sets *Mean-StD* and *Mean-CoV*. Each parameter has a *name*, e.g., the parameters of the set *Mean-StD* have the names *Mean* and *Stand-Dev.*

In Proban a parameter can be assigned a random variable or a numeric value (a number or a type fixed variable). In the latter case a parametric sensitivity factor can be calculated.

A distribution may be *multivariate*. In this case the dimension of the distribution will be required as input. Note that the dimension is *not* considered a parameter of a distribution (parametric sensitivity cannot be calculated with respect to dimension).

The continuous distributions available are shown in Table .

Beta	Beta Distribution
Burr	Burr Distribution
Chi-Square	χ^2 -Distribution
Exponential	Exponential Distribution
Gen-Gamma	Generalized Gamma Distribution
Gen-Par	Generalized Pareto Distribution
Gumbel	Gumbel Distribution
Hermit-Secon	Hermite Transformation Model - Second Order
Hermit-Trans	Hermite Transformation Model - First Order

Inv-Gauss	Inverse Gauss Distribution
Lognormal	Lognormal Distribution
Long-Higgins	Longuet-Higgins Distribution
Maxwell	Maxwell distribution
Multinormal	Multivariate Normal Distribution
Normal	Normal Distribution
Onesi-Normal	One-Sided Normal Distribution
POD-Distrib	Probability Of Detection Distribution
Oval	Oval Distribution
Rayleigh	Rayleigh Distribution
Spline-1Dim	One Dimensional Splined Distribution
Student-t	Student-t Distribution
Triangle	Triangle Distribution
Trunc-Normal	Truncated Normal Distribution
Uniform	Uniform Distribution
Weibull	Weibull Distribution

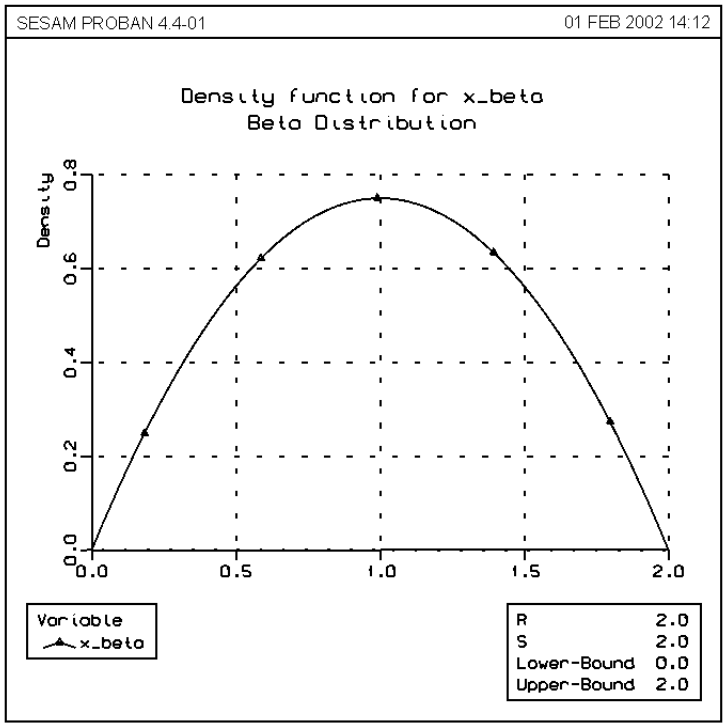
2.2 Beta Distribution

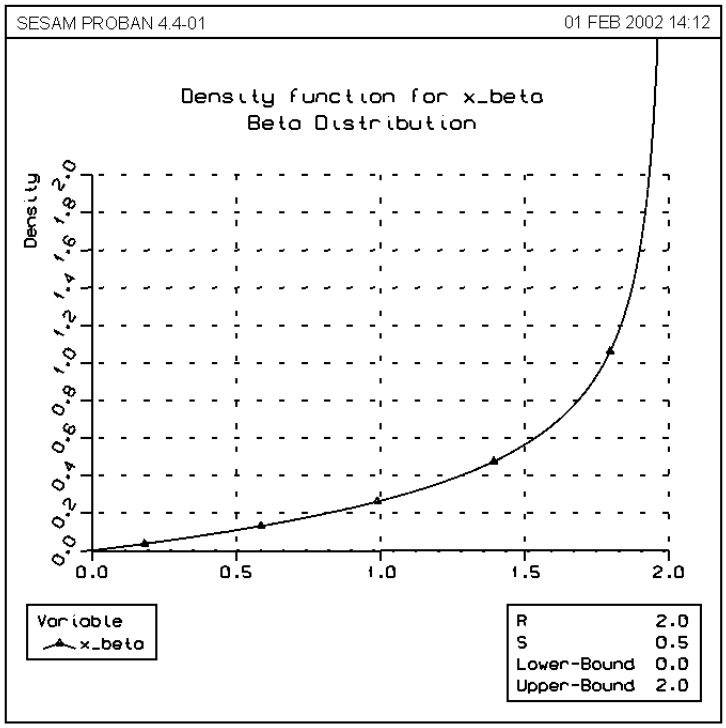
Name	Beta
Density	$f(x) = \frac{1}{(b-a)^{t-1}B(r, t-r)}(x-a)^{r-1}(b-x)^{t-r-1}$
Cumulative	$F(x) = \frac{1}{(b-a)^{t-1}B(r, t-r)}\int_a^x (v-a)^{r-1}(b-v)^{t-r-1}dv$
Restrictions	$a \le x \le b,\ r > 0,\ t > r$
s (parameter)	$s = t - r$
Lower Bound	a
Upper Bound	b
Moments	$E((X-a)^n) = \frac{\Gamma(r+n)\Gamma(t)}{\Gamma(r)\Gamma(t+1)}(b-a)^n$
Mean	$\mu = a + (b-a)\frac{r}{t}$
Standard Deviation	$\sigma = (b-a)\frac{r}{t}\frac{\sqrt{t-r}}{\sqrt{r(t+1)}}$
Skewness	$\delta = 2\frac{t-2r}{t+2}\frac{\sqrt{t+1}}{\sqrt{r(t-r)}}$
Kurtosis	$\kappa = \frac{3(t+1)((r+2)t^2-(r+6)t+6)}{r(t+2)(t+3)(t-r)}$

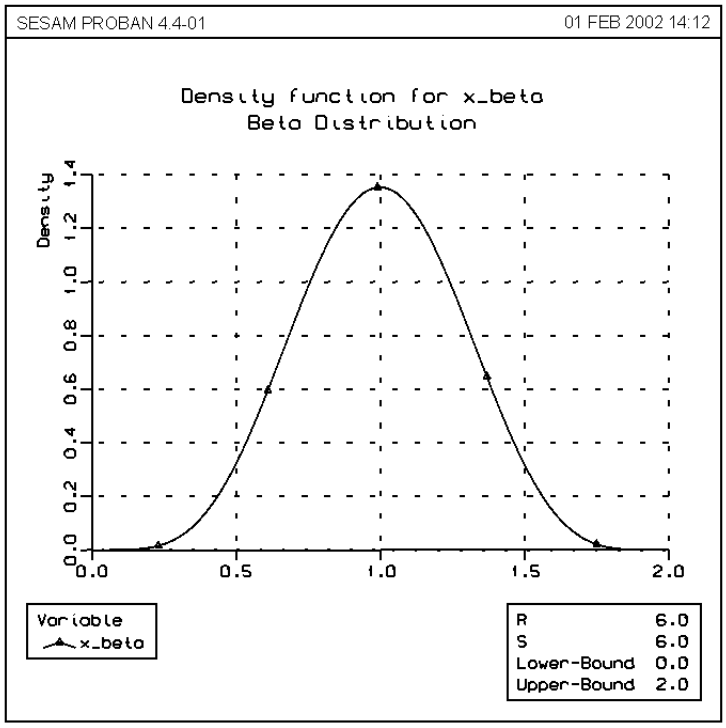
Parameter Set	Parameter Names	Symbol	Restriction
Mean-StD-Lim	Mean	μ	
	Stand-Dev	σ	> 0
	Lower-Bound	a	$< \mu$
	Upper-Bound	b	$> \mu$
Mean-CoV-Lim	Mean	μ	$\neq 0$
	Coef-of-Var	$\lvert\sigma\rvert/\mu$	> 0
	Lower-Bound	a	$< \mu$
	Upper-Bound	b	$> \mu$
R-T-Lim	R	r	> 0
	T	t	$> r$
	Lower-Bound	a	
	Upper-Bound	b	$> a$
R-S-Lim	R	r	
	S	s	> 0
	Lower-Bound	a	
	Upper-Bound	b	$> a$
Low-MostL-Up	Lower-Bound	a	$< ml$
	Most-Likely	ml	$\mu = \frac{a + 4ml + b}{6}, \sigma = \frac{b - a}{6}$
	Upper-Bound	b	$> ml$

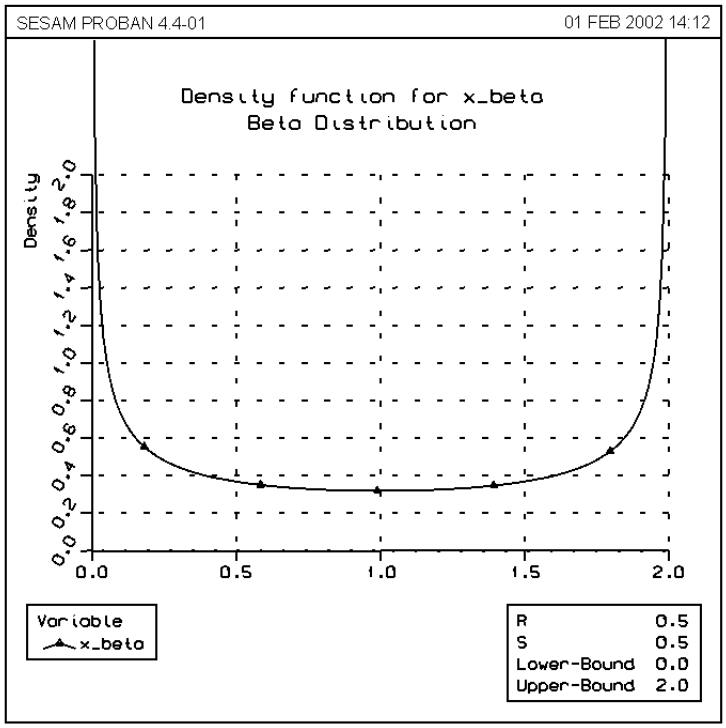
The Beta distribution is a flexible tool for modelling the distribution of a bounded random variable.

The “Most-Likely” parameter is not an exact most likely value for the distribution. It is used to calculate a mean and a standard deviation for the distribution as above.


SESAM PROBAN 4.4-01
01 FEB 2002 14:12
Density function for x_beta
Beta Distribution
Density
0.0
0.2
0.4
0.6
0.8
0.0
0.5
1.0
1.5
2.0
Variable
x_beta
R 2.0
S 2.0
Lower-Bound 0.0
Upper-Bound 2.0


SESAM PROBAN 4.4-01
01 FEB 2002 14:12
Density function for x_beta
Beta Distribution
Density
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
0.0
0.5
1.0
1.5
2.0
Variable
x_beta
R 2.0
S 0.5
Lower-Bound 0.0
Upper-Bound 2.0

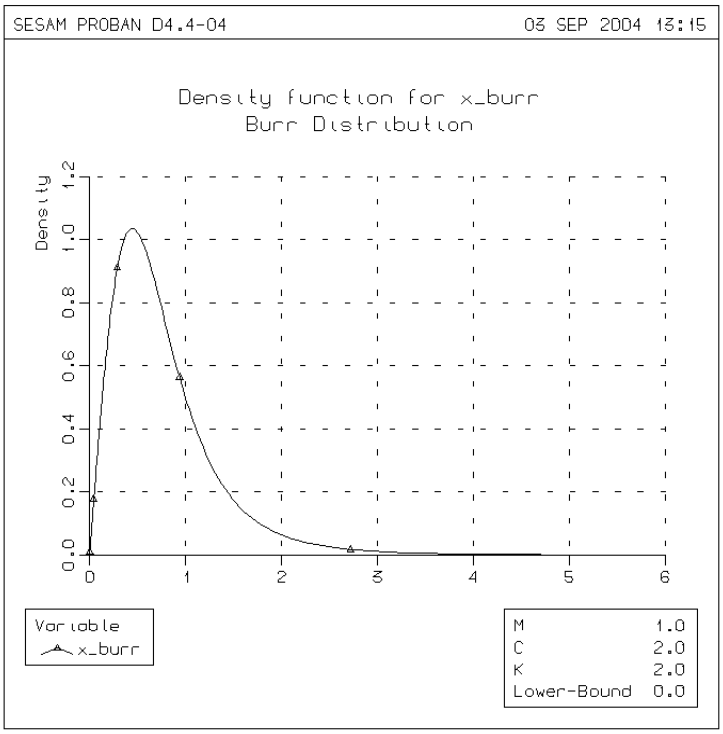

SESAM PROBAN 4.4-01
01 FEB 2002 14:12
Density function for x_beta
Beta Distribution
Density
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
0.0
0.5
1.0
1.5
2.0
Variable
x_beta
R 6.0
S 6.0
Lower-Bound 0.0
Upper-Bound 2.0

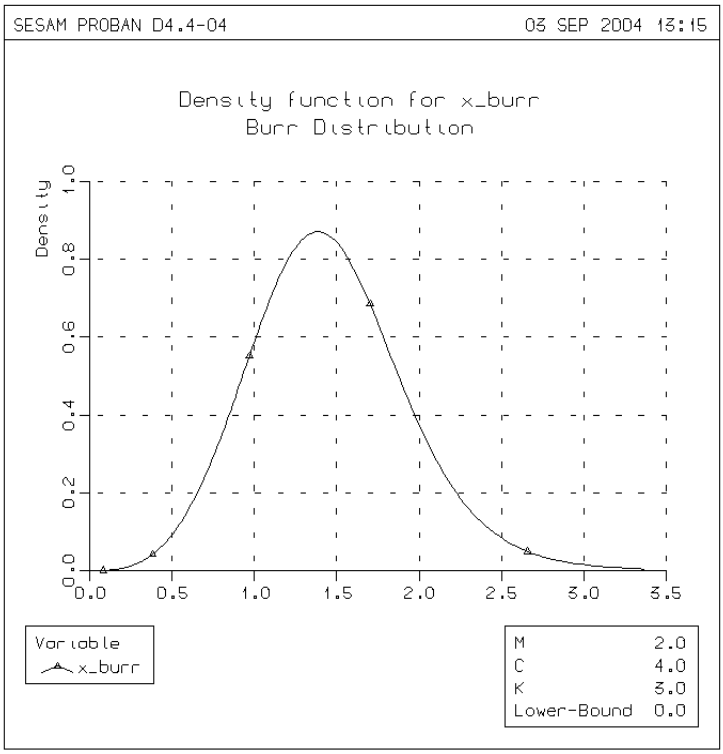

SESAM PROBAN 4.4-01
01 FEB 2002 14:12
Density function for x_beta
Beta Distribution
Density
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
0.0
0.5
1.0
1.5
2.0
Variable
x_beta
R 0.5
S 0.5
Lower-Bound 0.0
Upper-Bound 2.0

2.3 Burr Distribution

Name	Burr
Density	$f(x) = \frac{ck}{m-\gamma}\left(\frac{x-\gamma}{m-\gamma}\right)^{c-1}\left(1+\left(\frac{x-\gamma}{m-\gamma}\right)^{c}\right)^{-k-1}$
Cumulative	$F(x) = 1-\left(1+\left(\frac{x-\gamma}{m-\gamma}\right)^{c}\right)^{-k}$
Restrictions	$x \geq \gamma,\ m > 0,\ c > 0,\ k > 0$
Lower Bound	γ
Median	$(m-\gamma)(2^{1/k}-1)^{1/c}+\gamma$
Moments	$E((x-\gamma)^n) = (m-\gamma)^n B\left(1+\frac{n}{c}, k-\frac{n}{c}\right),\ k > \frac{n}{c}$, else ∞
Mean	$(m-\gamma)B\left(1+\frac{1}{c}, k-\frac{1}{c}\right)+\gamma, \quad c > \frac{1}{k}$
Standard Deviation	$(m-\gamma)\sqrt{B\left(1+\frac{2}{c}, k-\frac{2}{c}\right)-B\left(1+\frac{1}{c}, k-\frac{1}{c}\right)^2}, \quad c > \frac{2}{k}$, else ∞
Skewness	$\frac{B\left(1+\frac{3}{c}, k-\frac{3}{c}\right)-3B\left(1+\frac{1}{c}, k-\frac{1}{c}\right)B\left(1+\frac{2}{c}, k-\frac{2}{c}\right)+2B\left(1+\frac{1}{c}, k-\frac{1}{c}\right)^3}{\left(B\left(1+\frac{2}{c}, k-\frac{2}{c}\right)-B\left(1+\frac{1}{c}, k-\frac{1}{c}\right)^2\right)^{3/2}}, \quad c > \frac{3}{k}$
Kurtosis	$\frac{E((x-\gamma)^4)-4E(x-\gamma)(E((x-\gamma)^3))+6E(x-\gamma)^2(E((x-\gamma)^2))-3E(x-\gamma)^3}{(E((x-\gamma)^2)-E(x-\gamma)^2)^2}, c > \frac{4}{k}$

Parameter Set	Parameter Names	Symbol	Restriction
M-C-K-Low	M	m	$> \gamma$
	C	c	> 0
	K	k	> 0
	Lower-Bound	γ	


SESAM PROBAN D4.4-04
03 SEP 2004 13:15
Density function for x_burr
Burr Distribution
Density
0.0
0.2
0.4
0.6
0.8
1.0
1.2
0
1
2
3
4
5
6
Variable
x_burr
M 1.0
C 2.0
K 2.0
Lower-Bound 0.0


SESAM PROBAN D4.4-04
03 SEP 2004 13:15
Density function for x_burr
Burr Distribution
Density
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
Variable
x_burr
M 2.0
C 4.0
K 3.0
Lower-Bound 0.0

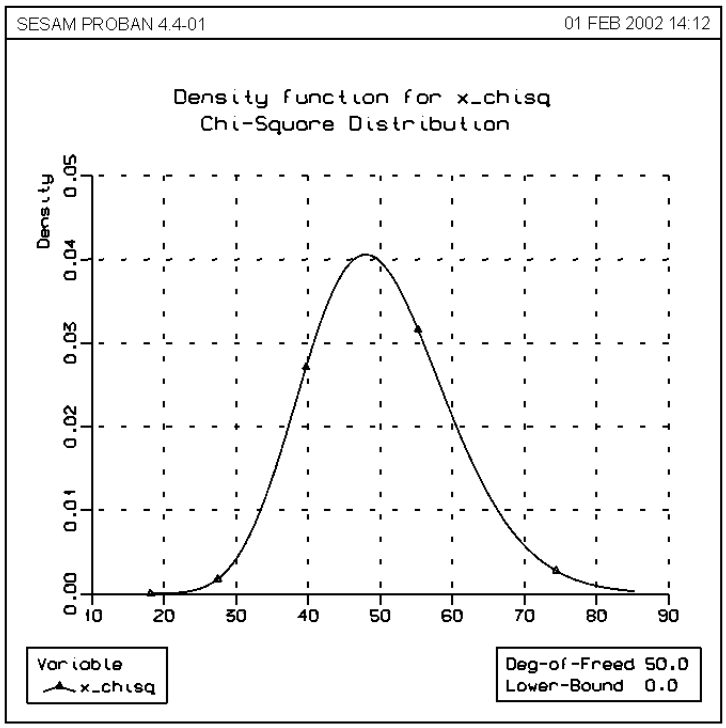
2.4 Chi-Square Distribution

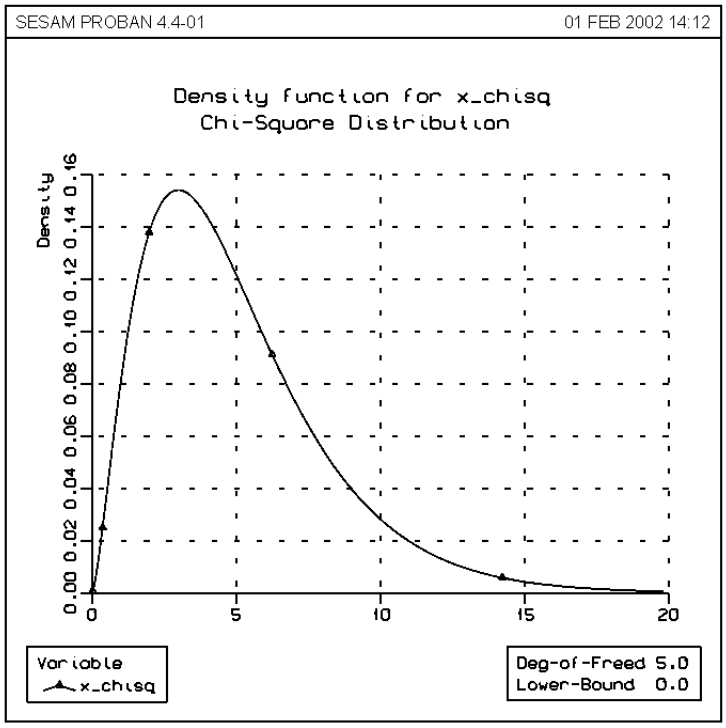
Name	Chi-Square
Density	$f(x) = \frac{1}{2\Gamma\left(\frac{\nu}{2}\right)}\left(\frac{x-\gamma}{2}\right)^{\frac{\nu}{2}-1} e^{-\frac{x-\gamma}{2}}$
Cumulative	$F(x) = \frac{\gamma\left(\frac{\nu}{2}, \frac{x-\gamma}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}$
Restrictions	$x \geq \gamma, \nu > 0$
Number of degrees of freedom	ν
Lower Bound	γ
Median	$2\gamma^{-1}\left(\frac{\nu}{2}, 0.5\Gamma\left(\frac{\nu}{2}\right)\right) + \gamma$
Moments	$E((X-\gamma)^n) = 2^n \frac{\Gamma\left(\frac{\nu}{2}+n\right)}{\Gamma\left(\frac{\nu}{2}\right)}$
Mean	$\mu = \nu + \gamma$
Standard Deviation	$\sigma = \sqrt{2\nu}$
Skewness	$\delta = \frac{1}{\sqrt{2\nu}}$
Kurtosis	$\kappa = 3 + \frac{12}{\nu}$

Parameter Set	Parameter Names	Symbol	Restriction
Mean-Low	Mean	μ	
	Lower-Bound	γ	$< \mu$
DoF-Low	Deg-of-Freed	ν	> 0
	Lower-Bound	γ	

The χ^2 distribution is the Generalized Gamma distribution with parameters a=ν/2, b=1, c=2

The χ^2 distribution is the Gamma distribution with parameters k=ν/2, λ=1/2.




SESAM PROBAN 4.4-01
01 FEB 2002 14:12
Density function for x_chisq
Chi-Square Distribution
Density
0.00 0.02 0.04 0.06 0.08 0.10 0.12 0.14 0.16
0
5
10
15
20
Variable
x_chisq
Deg-of-Freed 5.0
Lower-Bound 0.0

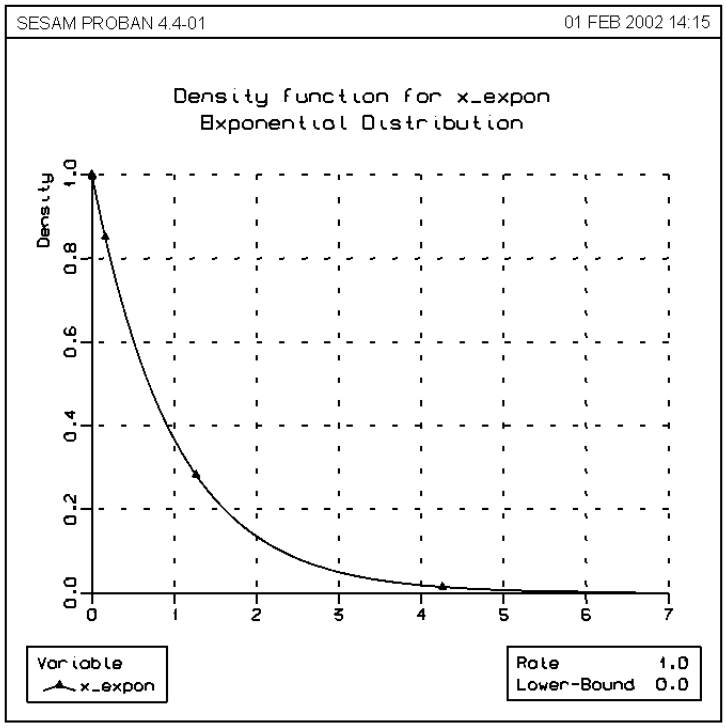
2.5 Exponential Distribution

Name	Exponential
Density	$f(x) = \lambda e^{-\lambda(x-\gamma)}$
Cumulative	$F(x) = 1 - e^{-\lambda(x-\gamma)}$
Restrictions	$x \geq \gamma, \lambda > 0$
Rate	λ
Lower Bound	γ
Median	$-\frac{\ln(0.5)}{\lambda} + \gamma$
Moments	$E((X-\gamma)^n) = n!\lambda^{-n}$
Mean	$\mu = \lambda^{-1} + \gamma$
Standard Deviation	$\sigma = \lambda^{-1}$
Skewness	$\delta = 2$
Kurtosis	$\kappa = 9$

Parameter Set	Parameter Names	Symbol	Restriction
Mean-Low	Mean	μ	
	Lower-Bound	γ	$< \mu$
Rate-Low	Rate	λ	> 0
	Lower-Bound	γ	

The Exponential distribution is the Generalized Gamma distribution with parameters a=1, b=1, c=1/λ.

The Exponential distribution is the Gamma distribution with parameters k=1.
The Exponential distribution is the Weibull distribution with parameters β=1 and δ_w=λ


SESAM PROBAN 4.4-01
01 FEB 2002 14:15
Density function for x_expon
Exponential Distribution
Density
1.0
0.8
0.6
0.4
0.2
0.0
0
1
2
3
4
5
6
7
Variable
x_expon
Rate 1.0
Lower-Bound 0.0

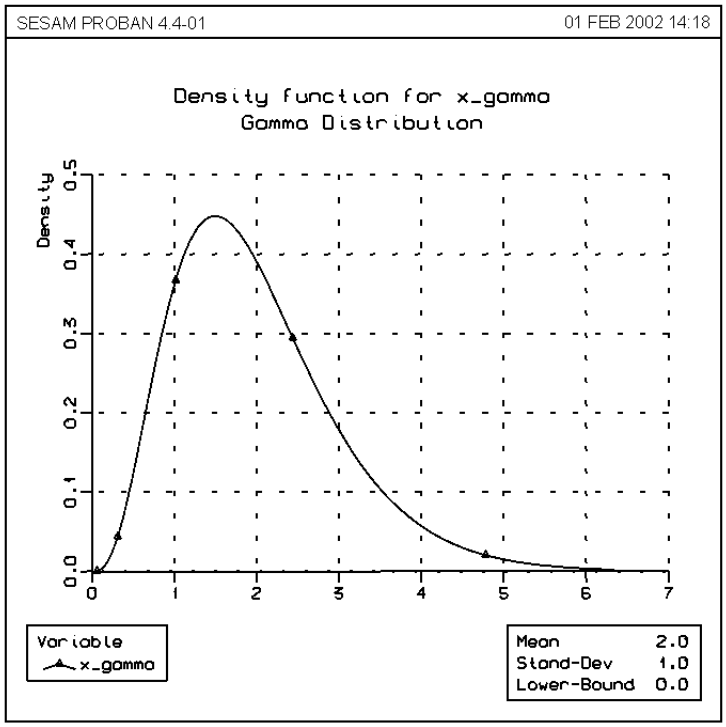
2.6 Gamma Distribution

Name	Gamma
Density	$f(x) = \frac{\lambda}{\Gamma(k)}(\lambda(x-\gamma))^{k-1}\exp(-\lambda(x-\gamma))$
Cumulative	$F(x) = \frac{\gamma(k,\lambda(x-\gamma))}{\Gamma(k)}$
Restrictions	$x \geq \gamma, k > 0, \lambda > 0$
Lower Bound	γ
Median	$\lambda^{-1}\gamma^{-1}(k, 0.5\Gamma(k)) + \gamma$
Moments	$E((X-\gamma)^n) = \lambda^{-n}\frac{\Gamma(k+n)}{\Gamma(k)}$
Mean	$\mu = k\lambda^{-1} + \gamma$
Standard Deviation	$\sigma = \lambda^{-1}\sqrt{k}$
Skewness	$\delta = \frac{2}{\sqrt{k}}$
Kurtosis	$\kappa = 3 + \frac{6}{k}$

Parameter Set	Parameter Names	Symbol	Restriction
Mean-StD-Low	Mean	μ	
	Stand-Dev	σ	> 0
	Lower-Bound	γ	$< \mu$
Mean-CoV-Low	Mean	μ	$\neq 0$
	Coef-of-Var	$\lvert\sigma\rvert/\mu$	> 0
	Lower-Bound	γ	$< \mu$
K-Lambda-Low	K	k	> 0
	Lambda	λ	> 0
	Lower-Bound	γ	

The Gamma distribution is the Generalized Gamma distribution with parameters $a=k$, $b=1$, and $c=1/\lambda$.

When *k* is an integer the distribution is also called the Erlang distribution.

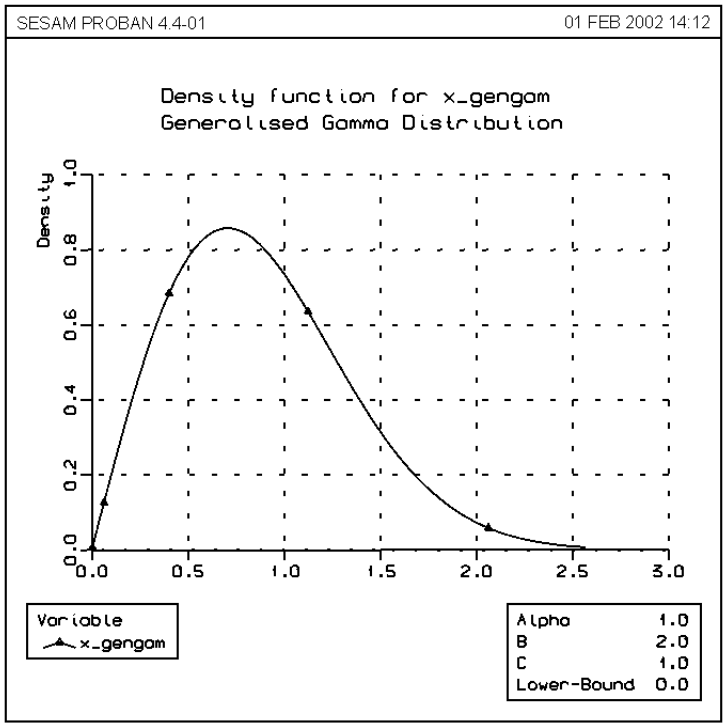


2.7 Generalized Gamma Distribution

Name	Gen-Gamma
Density	$f(x) = \frac{\lvert b\rvert}{c\Gamma(a)}\left(\frac{x-\gamma}{c}\right)^{ab-1}\exp\left(-\left(\frac{x-\gamma}{c}\right)^{b}\right)$
Cumulative, $b > 0$	$F(x) = \frac{\gamma\left(a,\left(\frac{x-\gamma}{c}\right)^{b}\right)}{\Gamma(a)}$
Cumulative, $b < 0$	$F(x) = \frac{\Gamma\left(a,\left(\frac{x-\gamma}{c}\right)^{b}\right)}{\Gamma(a)}$
Restrictions	$x \geq \gamma, a > 0, b \neq 0, c > 0$
Lower Bound	γ
Median	$c\gamma^{-1}(a, 0.5\Gamma(a))^{\frac{1}{b}} + \gamma$
Moments	$E((X-\gamma)^{n}) = c^{n}\frac{\Gamma\left(a+\frac{n}{b}\right)}{\Gamma(a)}$
Mean	$\mu = \frac{c}{\Gamma(a)}\Gamma\left(a+\frac{1}{b}\right) + \gamma$
Standard Deviation	$\sigma = \frac{c}{\Gamma(a)}\left\{\Gamma(a)\Gamma\left(a+\frac{2}{b}\right) - \Gamma\left(a+\frac{1}{b}\right)^{2}\right\}^{0.5}$
Skewness	$\delta = \frac{\Gamma(a)^{2}\Gamma\left(a+\frac{3}{b}\right) - 3\Gamma(a)\Gamma\left(a+\frac{1}{b}\right)\Gamma\left(a+\frac{2}{b}\right) + 2\Gamma\left(a+\frac{1}{b}\right)^{3}}{\left\{\Gamma(a)\Gamma\left(a+\frac{2}{b}\right) - \Gamma\left(a+\frac{1}{b}\right)^{2}\right\}^{1.5}}$

Kurtosis	$s = \Gamma(a)^3\Gamma\left(a+\frac{4}{b}\right) - 4\Gamma(a)^2\Gamma\left(a+\frac{1}{b}\right)\Gamma\left(a+\frac{3}{b}\right)$ $t = 6\Gamma(a)\Gamma\left(a+\frac{1}{b}\right)^2\Gamma\left(a+\frac{2}{b}\right) - 3\Gamma\left(a+\frac{1}{b}\right)^4$ $\kappa = \frac{s+t}{\left\{\Gamma(a)\Gamma\left(a+\frac{2}{b}\right) - \Gamma\left(a+\frac{1}{b}\right)^2\right\}^2}$

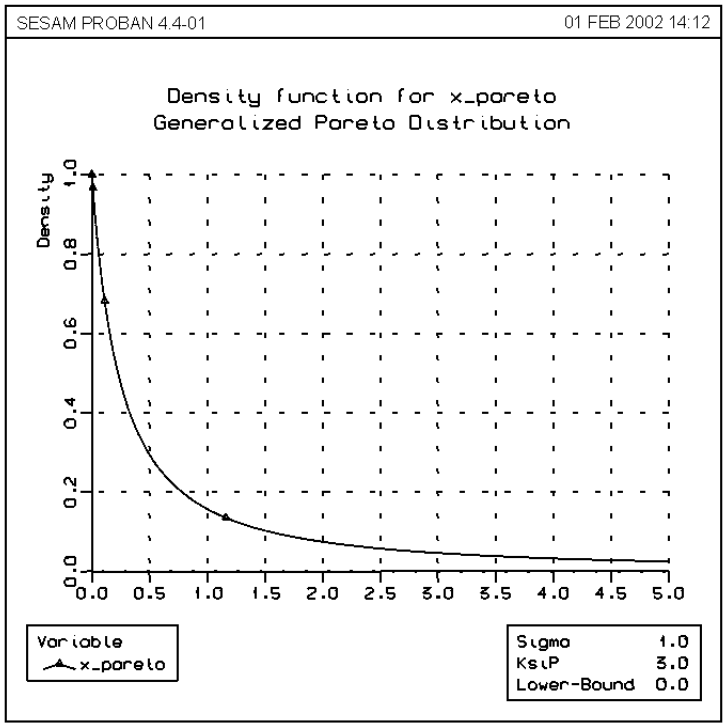
Parameter Set	Parameter Names	Symbol	Restriction
Al-B-C-Low	Alpha	a	> 0
	B	b	$\neq 0$
	C	c	> 0
	Lower-Bound	γ	



2.8 Generalized Pareto Distribution

Name	Gen-Pareto
Density	$f(x) = \frac{1}{\sigma}\left(1 + \frac{\xi(x-\gamma)}{\sigma}\right)^{-\frac{1}{\xi}-1}$
Cumulative	$F(x) = 1 - \left(1 + \frac{\xi(x-\gamma)}{\sigma}\right)^{-\frac{1}{\xi}}$
Restrictions	$x \geq \gamma, \xi > 0, \sigma > 0$
Lower Bound	γ
Median	$\frac{\sigma}{\xi}(0.5^{-\xi} - 1) + \gamma$
Moments	$E((x-\gamma)^n) = \frac{\sigma^n n!}{(1-\xi)(1-2\xi)\ldots(1-n\xi)}$
Mean	$\frac{\sigma}{1-\xi} + \gamma, \quad \xi < 1$
Standard Deviation	$\frac{\sigma}{(1-\xi)\sqrt{(1-2\xi)}}, \quad \xi < \frac{1}{2}$
Skewness	$\frac{2(1+\xi)\sqrt{1-2\xi}}{1-3\xi}, \quad \xi < \frac{1}{3}$
Kurtosis	$\frac{3(1-2\xi)(3+\xi+2\xi^2)}{(1-3\xi)(1-4\xi)}, \quad \xi < \frac{1}{4}$

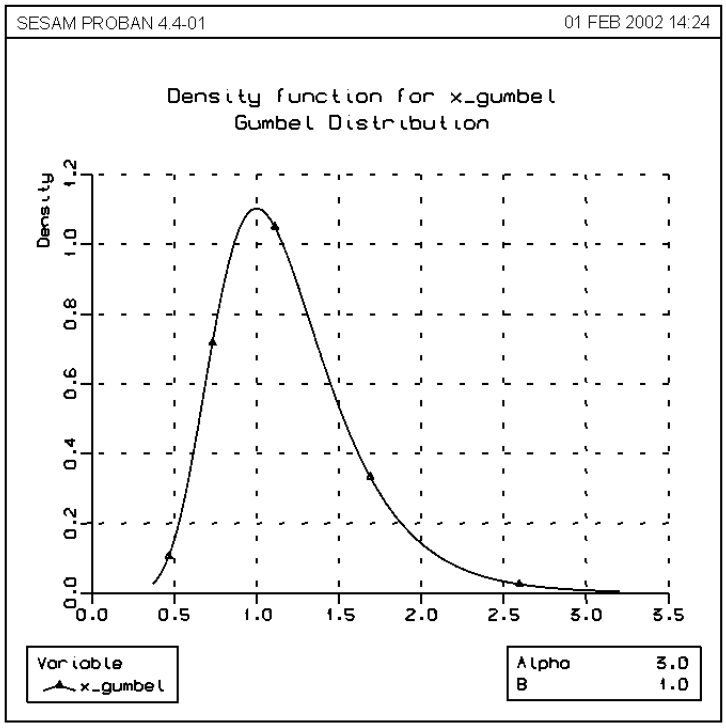
Parameter Set	Parameter Names	Symbol	Restriction
Sig-KsiP-Low	Sigma	σ	> 0
	KsiP	ξ	> 0
	Lower-Bound	γ	

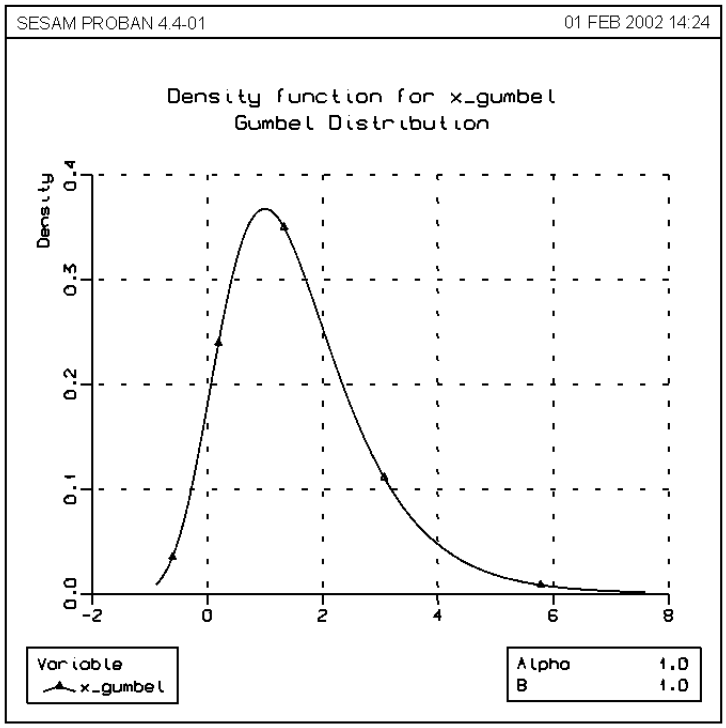

SESAM PROBAN 4.4-01
01 FEB 2002 14:12
Density function for x_pareto
Generalized Pareto Distribution
Density
0.0
0.2
0.4
0.6
0.8
1.0
0.0 0.5 1.0 1.5 2.0 2.5 3.0 3.5 4.0 4.5 5.0
Variable
x_pareto
Sigma 1.0
KsiP 3.0
Lower-Bound 0.0

2.9 Gumbel Distribution

Name	Gumbel
Density	$f(x) = \alpha \exp(-\alpha(x-\beta) - \exp(-\alpha(x-\beta)))$
Cumulative	$F(x) = \exp(-\exp(-\alpha(x-\beta)))$
Restrictions	$\alpha > 0$
Median	$-\frac{1}{\alpha}\ln(-\ln 0.5) + \beta$
Moments	$E((X-\beta)^n) = \left(\frac{-1}{\alpha}\right)^n \int_0^\infty \ln(s)^n e^{-s} ds$
Mean	$\mu = \frac{\gamma_e}{\alpha} + \beta$
Standard Deviation	$\sigma = \frac{\pi}{\sqrt{6}\alpha}$
Skewness	$\delta = -2\frac{\zeta(3)}{\zeta(2)^{1.5}} = -1.13954710$
Kurtosis	$\kappa = 3 + 6\frac{\zeta(4)}{\zeta(2)^2} = 5.40$

Parameter Set	Parameter Names	Symbol	Restriction
Mean-StD	Mean	μ	
	Stand-Dev	σ	> 0
Mean-CoV	Mean	μ	$\neq 0$
	Coef-of-Var	$\sigma / \lvert\mu\rvert$	> 0
Alpha-B	Alpha	α	> 0
	B	β	


SESAM PROBAN 4.4-01
01 FEB 2002 14:24
Density function for x_gumbel
Gumbel Distribution
Density
0.0
0.2
0.4
0.6
0.8
1.0
1.2
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
Variable
x_gumbel
Alpha 3.0
B 1.0


SESAM PROBAN 4.4-01
01 FEB 2002 14:24
Density function for x_gumbel
Gumbel Distribution
Density
0.0
0.1
0.2
0.3
0.4
-2
0
2
4
6
8
Variable
x_gumbel
Alpha 1.0
B 1.0

2.10 Hermite Transformation Model - Second Order

Name	Hermit-Second
Density	$f(x) = \phi(u(x))$
Cumulative	$F(x) = \Phi(u(x))$
$\kappa_i > 3$	$x(u) = \mu_i + \alpha\sigma_i(u + c_3(u^2 - 1) + c_4(u^3 - 3u))$ $c_4 = \frac{\sqrt{6\kappa_i - 14} - 2}{36}$ $c_3 = \frac{\delta_i}{6(1 + 6c_4)}$ $\alpha = \sqrt{1 + 2c_3^2 + 6c_4^2}$
$\kappa_i < 3$	$u(x) = y - c_3(y^2 - 1) - c_4(y^3 - 3y)$ $y = 6c_3c_4 + \frac{x - \mu_i}{\alpha\sigma_i}$ $c_4 = \frac{(\kappa_i - 3)(35 - 9\kappa_i)}{192}$ $c_3 = \frac{\delta_i}{6\lvert 1 + 24c_4\rvert}$ $\alpha = \sqrt{1 + 10c_3^2 + 42c_4^2}$
Restrictions	$\sigma_i > 0, \kappa_i > 0$
Limit for all distributions	$\delta_i^2 < \frac{8}{9}\kappa_i$
Limit for the monotonity of the transformation when $\kappa_i < 3$.	$\delta_i^2 \le -(81h_4^2 + 27h_4)(1 + h_4)^2$; $h_4 = \frac{1}{8}(\kappa_i - 3)(35 - 9\kappa_i)$
Limit for the monotonity of the transformation when $\kappa_i \ge 3$.	$h_3^2 < 3h_4(1 - 3h_4)$; $h_3 = \frac{\delta_i}{6(1 + 6h_4)}$; $h_4 = \frac{1}{18}(\sqrt{1 + 1.5(\kappa_i - 3)} - 1)$
Input Moments	$\mu_i, \sigma_i, \delta_i, \kappa_i$

Median	$m = u^{-1}(0)$
Mean	μ
Standard deviation	$\sigma = \sqrt{E((x-\mu)^2)}$
Skewness	$\delta = \frac{E((X-\mu)^3)}{\sigma^3}$
Kurtosis	$\kappa = \frac{E((X-\mu)^4)}{\sigma^4}$

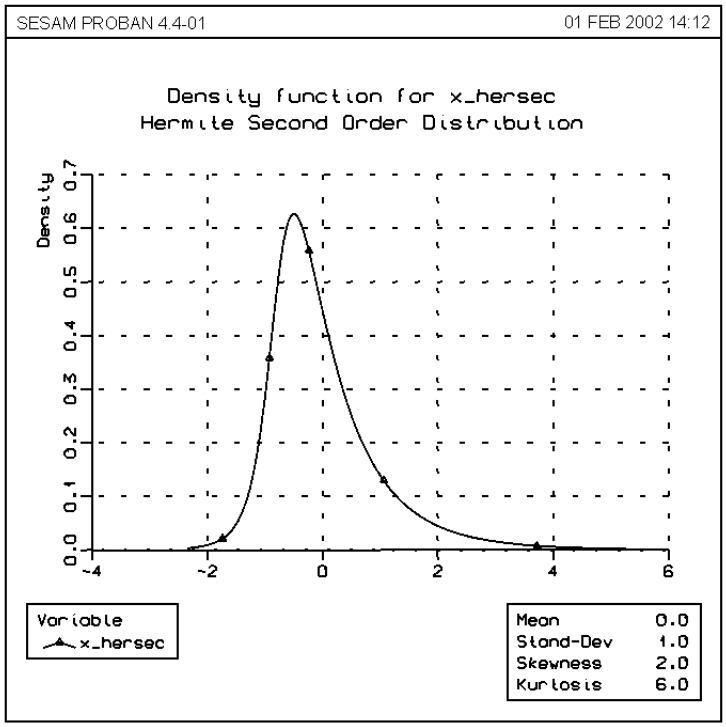
Parameter Set	Parameter Names	Symbol	Restriction
Moments	Mean	μ_i	
	Stand-Dev	σ_i	> 0
	Skewness	δ_i	$\delta_i * \delta_i \leq (8/9)\ \kappa_i$
	Kurtosis	κ_i	> 0
Central-Mom	Mean	μ_i	
	Variance	σ_i^2	> 0
	Third-C-Mom	$\delta_i \sigma_i^3$	
	Fourth-C-Mom	$\kappa_i \sigma_i^4$	> 0

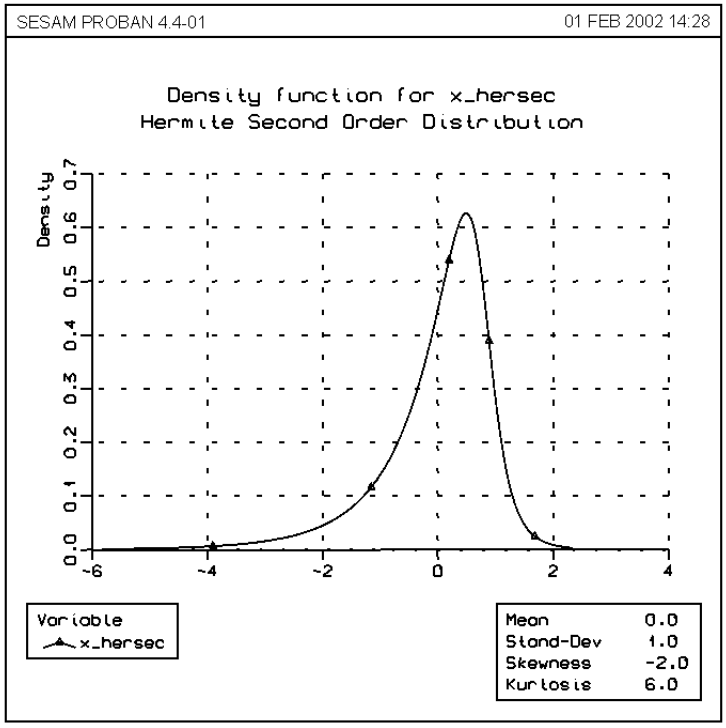
The Hermite Second Order Model yields a distribution with moments, μ, σ, δ, κ, which are close to, but not identical, the corresponding input moments μ_i, σ_i, δ_i, κ_i.

The Hermite Second Order Model is identical to the Hermite First Order Model except for $\kappa_i < 3$ where it yields moments closer to the input moments than the latter model does.

Winterstein,S.R.,"Non-normal Responses and Fatigue damage", *Journal of Engineering Mechanics*, ASCE, Vol 111, No. 10, Oct 1985, pp. 1291-1295.

Winterstein,S.R.,"Nonlinear Vibration Models for Extremes and Fatigue", *Journal of Engineering Mechanics*, ASCE, Vol 114, No. 10, Oct 1988, pp. 1772-1790.


SESAM PROBAN 4.4-01
01 FEB 2002 14:12
Density function for x_hersec
Hermite Second Order Distribution
Density
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7
-4 -2 0 2 4 6
Variable
x_hersec
Mean 0.0
Stand-Dev 1.0
Skewness 2.0
Kurtosis 6.0


SESAM PROBAN 4.4-01
01 FEB 2002 14:28
Density function for x_hersec
Hermite Second Order Distribution
Density
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7
-6 -4 -2 0 2 4
Variable
x_hersec
Mean 0.0
Stand-Dev 1.0
Skewness -2.0
Kurtosis 6.0

2.11 Hermite Transformation Model - First Order

Name	Hermit-Trans
Density	$f(x) = \phi(u(x))$
Cumulative	$F(x) = \Phi(u(x))$
$\kappa_i > 3$	$x(u) = \mu_i + \alpha\sigma_i(u + c_3(u^2 - 1) + c_4(u^3 - 3u))$ $c_4 = \frac{\sqrt{6\kappa_i - 14} - 2}{36}$ $c_3 = \frac{\delta_i}{6(1 + 6c_4)}$ $\alpha = \sqrt{1 + 2c_3^2 + 6c_4^2}$
$\kappa_i < 3$	$u(x) = y - c_3(y^2 - 1) - c_4(y^3 - 3y)$ $y = \left(\frac{x - \mu_i}{\sigma_i}\right)$ $c_4 = \frac{(\kappa_i - 3)}{24}$ $c_3 = \frac{\delta_i}{6}$
Restrictions	$\sigma_i > 0, \kappa_i > 0$
Limit for all distributions	$\delta_i^2 < \frac{8}{9}\kappa_i$
Limit for the monotonity of the transformation when $\kappa_i < 3$.	$\delta_i^2 \le \frac{9}{16}(3 - \kappa_i)(5 + \kappa_i)$
Limit for the monotonity of the transformation when $\kappa_i \ge 3$.	$h_3^2 < 3h_4(1 - 3h_4)$ $h_3 = \frac{\delta_i}{6(1 + 6h_4)}$ $h_4 = \frac{1}{18}(\sqrt{1 + 1.5(\kappa_i - 3)} - 1)$

Input Moments	$\mu_i, \sigma_i, \delta_i, \kappa_i$
Median	$m = u^{-1}(0)$
Mean	μ
Standard Deviation	$\sigma = \sqrt{E((x-\mu)^2)}$
Skewness	$\delta = \frac{E((X-\mu)^3)}{\sigma^3}$
Kurtosis	$\kappa = \frac{E((X-\mu)^4)}{\sigma^4}$

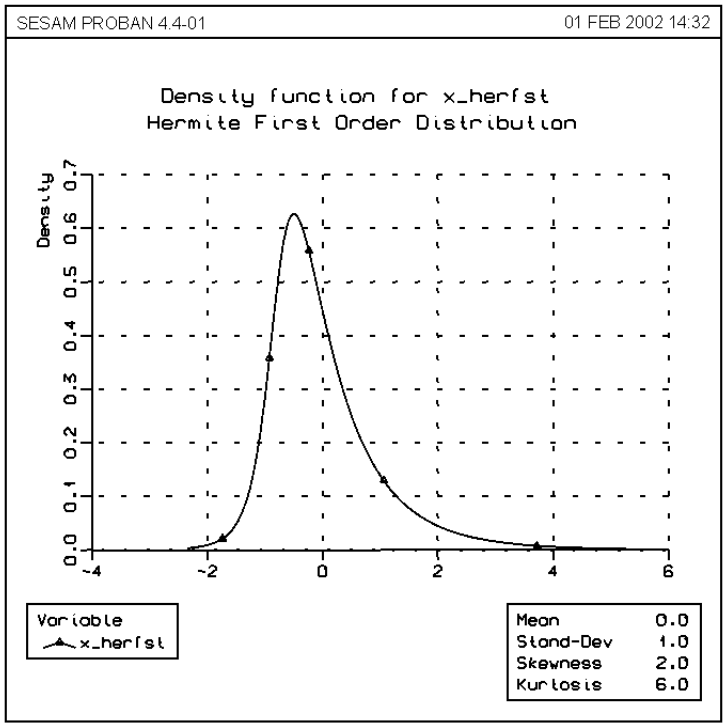
Parameter Set	Parameter Names	Symbol	Restriction
Moments	Mean	μ_i	
	Stand-Dev	σ_i	> 0
	Skewness	δ_i	$\delta_i * \delta_i \leq (8/9)\ \kappa_i$
	Kurtosis	κ_i	> 0
Central-Mom	Mean	μ_i	
	Variance	σ_i^2	> 0
	Third-C-Mom	$\delta_i \sigma_i^3$	
	Fourth-C-Mom	$\kappa_i \sigma_i^4$	> 0

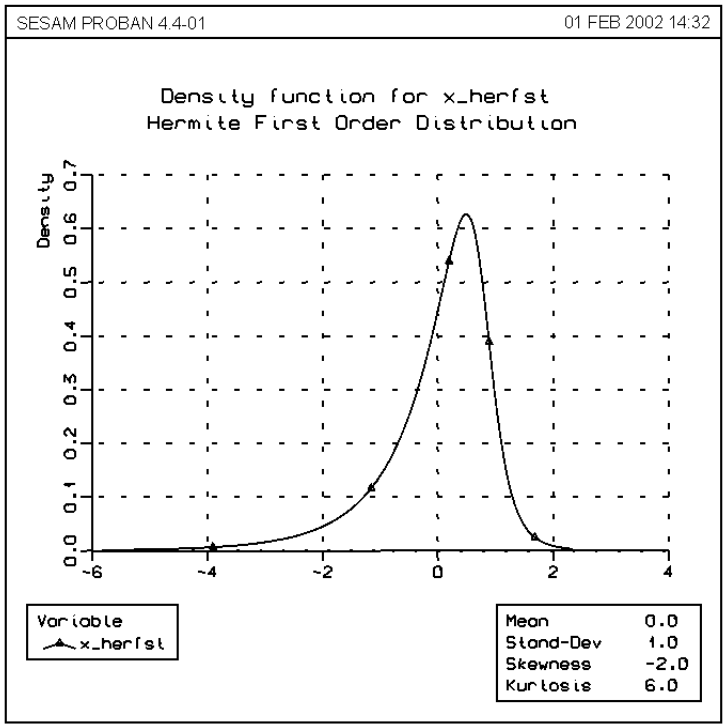
The Hermite First Order Model yields a distribution with moments, μ, σ, δ, κ, which are close to, but not identical, the corresponding input moments μ_i, σ_i, δ_i, κ_i.

The model uses response moments to construct non-gaussian contributions, made orthogonal through a Hermite series. The results are found to compare well with exact results for various nonlinear models, including nonlinear oscillator responses and quasi-static responses to non-Gaussian (Morison) wave loads.

Winterstein,S.R.,“Non-normal Responses and Fatigue damage”, *Journal of Engineering Mechanics*, ASCE, Vol 111, No. 10, Oct 1985, pp. 1291-1295.

Winterstein,S.R.,“Nonlinear Vibration Models for Extremes and Fatigue”, *Journal of Engineering Mechanics*, ASCE, Vol 114, No. 10, Oct 1988, pp. 1772-1790.

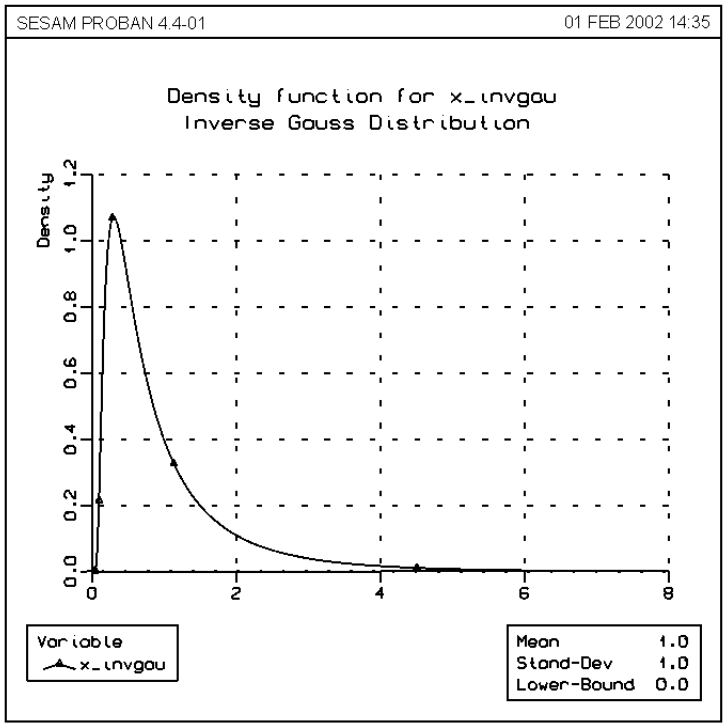

SESAM PROBAN 4.4-01
01 FEB 2002 14:32
Density function for x_herfst
Hermite First Order Distribution
Density
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7
-4 -2 0 2 4 6
Variable
x_herfst
Mean 0.0
Stand-Dev 1.0
Skewness 2.0
Kurtosis 6.0


SESAM PROBAN 4.4-01
01 FEB 2002 14:32
Density function for x_herfst
Hermite First Order Distribution
Density
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7
-6 -4 -2 0 2 4
Variable
x_herfst
Mean 0.0
Stand-Dev 1.0
Skewness -2.0
Kurtosis 6.0

2.12 Inverse Gauss Distribution

Name	Inv-Gauss
Density	$f(x) = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\lambda}}{\sqrt{(x-\gamma)^3}} \exp\left(-\frac{\lambda(x-\gamma-\xi)^2}{2\xi^2(x-\gamma)}\right)$
Cumulative	$F(x) = \Phi\left(\frac{\sqrt{\lambda(x-\gamma)}}{\xi} - \frac{\sqrt{\lambda}}{\sqrt{x-\gamma}}\right) + \Phi\left(-\frac{\sqrt{\lambda(x-\gamma)}}{\xi} - \frac{\sqrt{\lambda}}{\sqrt{x-\gamma}}\right) \exp\left(2\frac{\lambda}{\xi}\right)$
Restrictions	$\xi > 0, \lambda > 0$
Median	$m = F^{-1}(0.5)$
Mean	$\mu = \xi + \gamma$
Standard Deviation	$\sigma = \frac{\xi\sqrt{\xi}}{\sqrt{\lambda}}$
Skewness	$\delta = 3\frac{\sqrt{\xi}}{\sqrt{\lambda}}$
Kurtosis	$\kappa = 15\frac{\xi}{\lambda} + 3$

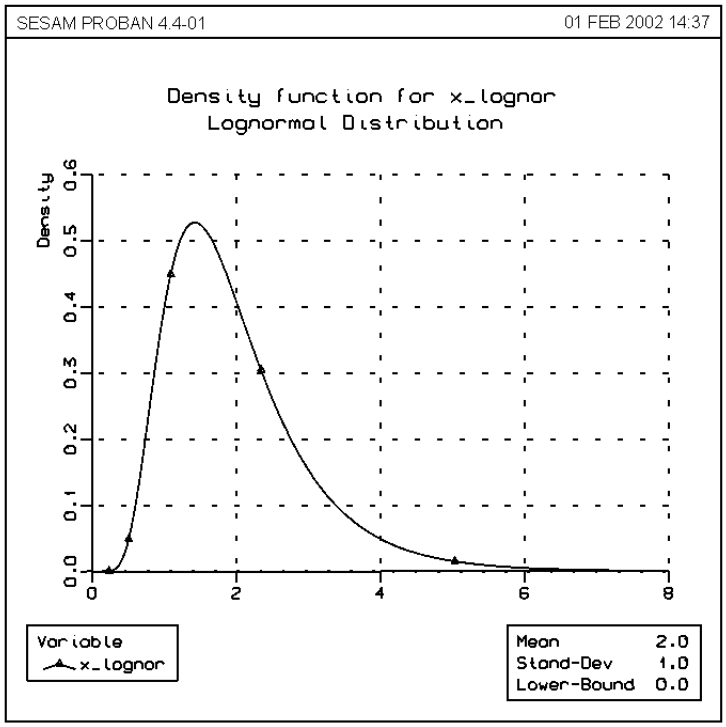
Parameter Set	Parameter Names	Symbol	Restriction
Mean-StD-Low	Mean	μ	
	Stand-Dev	σ	> 0
	Lower-Bound	γ	
Mean-CoV-Low	Mean	μ	$\neq 0$
	Coef-of-Var	$\sigma/\lvert\mu\rvert$	> 0
	Lower-Bound	γ	
Ksi-Lamb-Low	Ksi	ξ	> 0
	Lambda	λ	> 0
	Lower-Bound	γ	


SESAM PROBAN 4.4-01
01 FEB 2002 14:35
Density function for x_invgau
Inverse Gauss Distribution
Density
0.0
0.2
0.4
0.6
0.8
1.0
1.2
0
2
4
6
8
Variable
x_invgau
Mean 1.0
Stand-Dev 1.0
Lower-Bound 0.0

2.13 Lognormal Distribution

Name	Lognormal
Density	$f(x) = \frac{1}{\sqrt{2\pi}\sigma_l(x-\gamma)} \exp\left(-\frac{1}{2}\left(\frac{\ln(x-\gamma)-\mu_l}{\sigma_l}\right)^2\right)$
Cumulative	$F(x) = \frac{1}{\sqrt{2\pi}} \int_{\gamma}^{x} \frac{1}{\sigma_l(s-\gamma)} \exp\left(-\frac{1}{2}\left(\frac{\ln(s-\gamma)-\mu_l}{\sigma_l}\right)^2\right) ds$
Relation to Standard Normal Distribution	$F(x) = \Phi\left(\frac{\ln(x-\gamma_l)-\mu_l}{\sigma_l}\right)$
Restrictions	$x \geq 0, \sigma_l > 0$
Median	$\exp(\mu_l) + \gamma$
Moments	$E((x-\gamma)^n) = \exp\left(\frac{(n\sigma_l)^2}{2} + n\mu_l\right)$
Mean	$\mu = \exp\left(\frac{\sigma_l^2}{2} + \mu_l\right) + \gamma$
Standard Deviation	$\sigma = \exp\left(\frac{\sigma_l^2}{2} + \mu_l\right)(\exp(\sigma_l^2) - 1)^{0.5}$
Skewness	$\delta = (\exp(\sigma_l^2) - 1)^{0.5}(\exp(\sigma_l^2) - 2)$
Kurtosis	$\kappa = \exp(4\sigma_l^2) + 2\exp(3\sigma_l^2) + 3\exp(2\sigma_l^2) - 3$

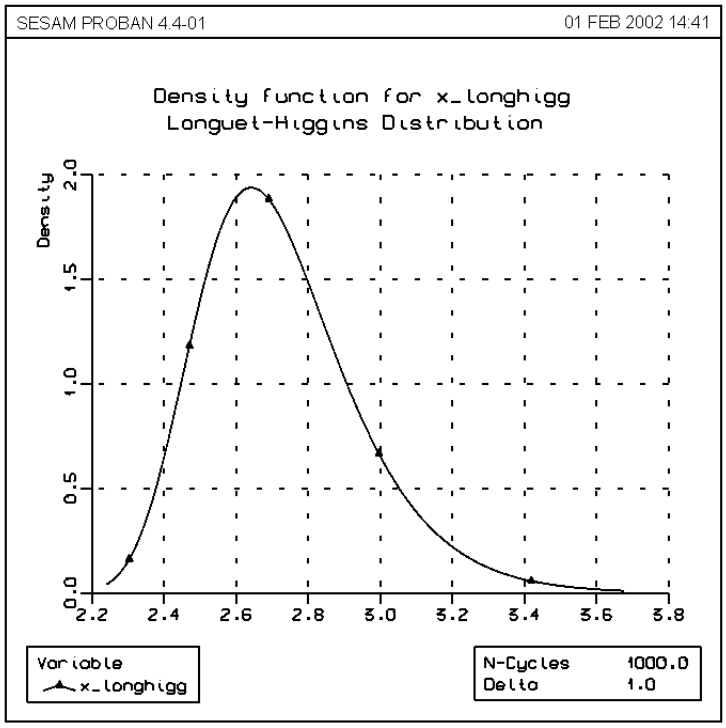
Parameter Set	Parameter Names	Symbol	Restriction
Mean-StD-Low	Mean	μ	
	Stand-Dev	σ	> 0
	Lower-Bound	γ	$< \mu$
Mean-CoV-Low	Mean	μ	$\neq 0$
	Coef-of-Var	$\sigma/\lvert\mu\rvert$	> 0
	Lower-Bound	γ	$< \mu$
Sigma-Lamb-Low	Sigma	σ_l	> 0
	Mu	μ_l	> 0
	Lower-Bound	γ	

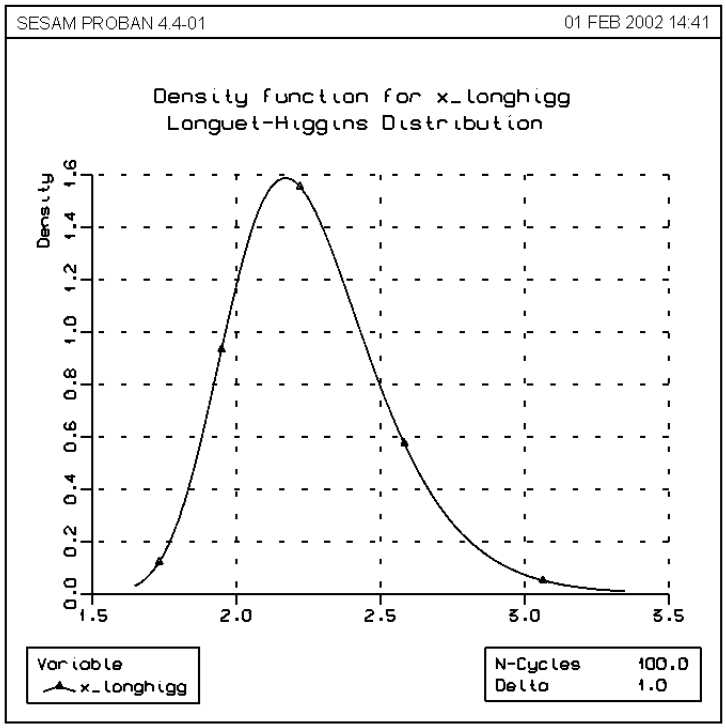


2.14 Longuet-Higgins Distribution

Name	Long-Higgins
Density	$f(x) = \dfrac{\frac{2Nx}{\delta_{lh}^2}\exp\left(-\left(\frac{x}{\delta_{lh}}\right)^2 - N\exp\left(-\left(\frac{x}{\delta_{lh}}\right)^2\right)\right)}{1-\exp(-N)}$
Cumulative	$F(x) = \dfrac{\exp\left(-N\exp\left(-\left(\frac{x}{\delta_{lh}}\right)^2\right)\right) - \exp(-N)}{1-\exp(-N)}$
Restrictions	$x \geq 0, N > 0, \delta_{lh} > 0$
Number of cycles	N
Median	$\delta_{lh}\left(-\ln\left(-\frac{1}{N}\ln(0.5(1+\exp(-N)))\right)\right)^{0.5}$
Moments	$E((x)^n) = \dfrac{\delta_{lh}^n(\ln(N))^{\frac{n}{2}}}{1-\exp(-N)}\int_0^N\left(1-\dfrac{\ln(s)}{\ln(N)}\right)^{\frac{n}{2}}\exp(-s)ds$
Mean	$\mu \sim \delta_{lh}\sqrt{\ln(N)} + \dfrac{\delta_{lh}\gamma_e}{\sqrt{\ln(N)}}, N \sim \infty$
Standard Deviation	$\sigma \sim \dfrac{\pi\delta_{lh}}{\sqrt{6}(\gamma_e + 2\ln(N))}, N \sim \infty$

Parameter Set	Parameter Names	Symbol	Restriction
NCycle-Delta	N-Cycles	N	> 0
	Delta	δ_{lh}	> 0


SESAM PROBAN 4.4-01
01 FEB 2002 14:41
Density function for x_longhigg
Longuet-Higgins Distribution
Density
0.0
0.5
1.0
1.5
2.0
2.2
2.4
2.6
2.8
3.0
3.2
3.4
3.6
3.8
Variable
x_longhigg
N-Cycles 1000.0
Delta 1.0

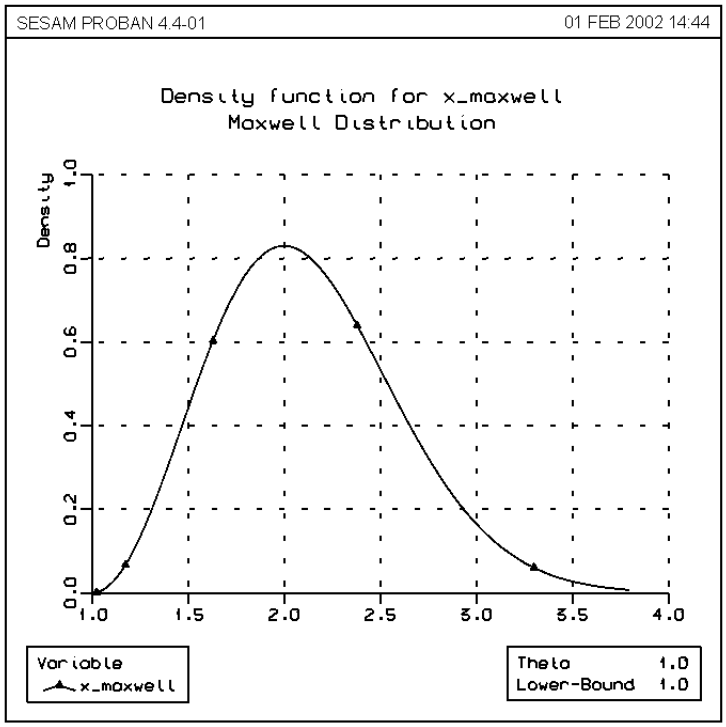

SESAM PROBAN 4.4-01
01 FEB 2002 14:41
Density function for x_longhigg
Longuet-Higgins Distribution
Density
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.5
2.0
2.5
3.0
3.5
Variable
x_longhigg
N-Cycles 100.0
Delta 1.0

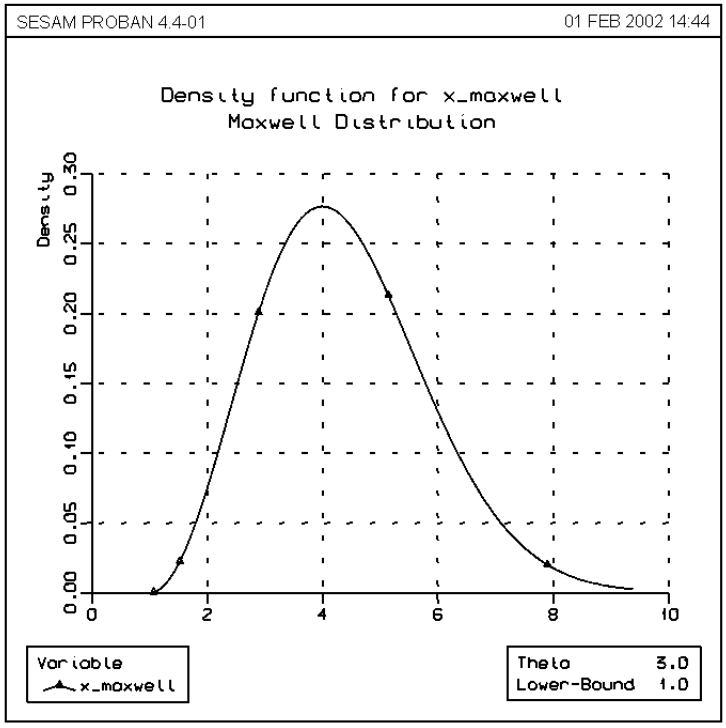
2.15 Maxwell Distribution

Name	Maxwell
Density	$f(x) = \frac{4}{\sqrt{\pi}\theta}\left(\frac{x-\gamma}{\theta}\right)^2 \exp\left(-\left(\frac{x-\gamma}{\theta}\right)^2\right)$
Cumulative	$F(x) = \frac{\gamma\left(\frac{3}{2}, \left(\frac{x-\gamma}{\theta}\right)^2\right)}{\Gamma\left(\frac{3}{2}\right)}$
Restrictions	$x \geq \gamma,\ \theta > 0$
Median	$m = \theta\Gamma^{-1}\left(\frac{3}{2}, \frac{\sqrt{\pi}}{4}\right)^{0.5} + \gamma$
Moments	$E((x-\gamma)^n) = \theta^n \frac{\Gamma\left(\frac{3}{2}+\frac{n}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}$
Mean	$\mu = \frac{2\theta}{\sqrt{\pi}} + \gamma$
Standard Deviation	$\sigma = \left(\frac{3}{2} - \frac{4}{\pi}\right)^{0.5}\theta = 0.476190\ \theta$
Skewness	$\delta = \frac{16 - 5\pi}{\left(\frac{3}{2} - \frac{4}{\pi}\right)^{1.5}} = 0.48569$
Kurtosis	$\kappa = \frac{15\pi^2 + 16\pi - 192}{(3\pi - 8)^2} = 3.10816$

Parameter Set	Parameter Names	Symbol	Restriction
Mean-Low	Mean	μ	
	Lower-Bound	γ	$< \mu$
Theta-Low	Theta	θ	> 0
	Lower-Bound	γ	

The Maxwell distribution is the Generalized Gamma distribution with parameters a=1.5, b=2 and c=θ.





2.16 Multinormal Distribution

Name	Multinormal
Density	$\phi(\boldsymbol{x};\boldsymbol{C}) = \dfrac{1}{(2\pi)^{\frac{n}{2}} det(\boldsymbol{C})^{\frac{1}{2}}} \exp(-(\boldsymbol{x}-\mu)^{\mathrm{T}} \boldsymbol{C}^{-1} (\boldsymbol{x}-\mu))$
Cumulative	$\Phi(\boldsymbol{x};\boldsymbol{C}) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \dots \int_{-\infty}^{x_n} \phi(\boldsymbol{s};\boldsymbol{C}) ds_1 ds_2 \dots ds_n$
Random variables	$\boldsymbol{X} = (X_1, X_2, \dots, X_n)$
Dimension	n
Covariance Matrix	$\boldsymbol{C}$
Mean = Median	$\mu = (\mu_1, \mu_2, \dots, \mu_n)$
Standard Deviations	$\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n) = (C_{11}^{0.5}, C_{22}^{0.5}, \dots, C_{nn}^{0.5})$
Correlation Matrix	$\boldsymbol{R} = \Sigma^{-1} \boldsymbol{C} \Sigma^{-1}$, $\Sigma_{ii} = \sigma_i$, $\Sigma_{ij} = 0$, $i \neq j$
Restrictions	$\boldsymbol{C}$ and $\boldsymbol{R}$ must be positive definite.

Input of Dimension	
Proban:	Dimension
User:	Gives $n \geq 1$

Parameter Set	Parameter Names	Symbol	Examples
Cor-StD-Low	Corr-i-j	R_{ij}	Corr-1-2, Corr-1-3,...,Corr-2-3, Corr-2-4,...
	StDev-i	σ_i	StDev-1, StDev-2,...
	Mean-i	μ_i	Mean-1, Mean-2,...
Cor-StD-Low	Covar-i-j	C_{ij}	Covar-1-1, Covar-1-2,...,Covar-2-2, Covar-2-3,...
	Mean-i	μ_i	Mean-1, Mean-2,...

The indices *i* and *j* of the parameters takes the following values:

In the case of a correlation matrix, *j* runs from *i*+1 to *n* and *i* runs from 1 to *n*.

In the case of a covariance matrix, *j* runs from *i* to *n* and *i* runs from *i* to *n*.

In the case of mean values and standard deviations, *i* runs from 1 to *n*.

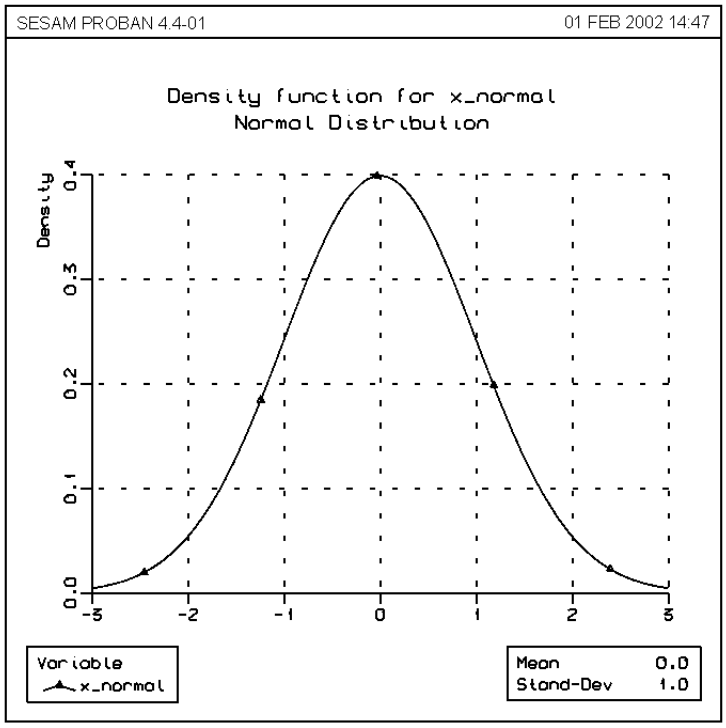
The Multinormal distribution may also be input by using the Nataf correlation model, see Section 4.3.

2.17 Normal Distribution

Name	Normal
Density	$f(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$
Cumulative	$F(x) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x}\exp\left(-\frac{1}{2}\left(\frac{s-\mu}{\sigma}\right)^2\right)ds$
Restrictions	$\sigma > 0$
Median	$m = \mu$
Moments	$E((x-\mu)^n) = \frac{2^{\frac{n+1}{2}}\sigma^n}{\sqrt{2\pi}}\Gamma\left(\frac{n+1}{2}\right)$; if n even, else 0.
Mean	μ
Standard Deviation	σ
Skewness	$\delta = 0$
Kurtosis	$\kappa = 3$

Parameter Set	Parameter Names	Symbol	Restriction
Mean-StD	Mean	μ	
	Stand-Dev	σ	> 0
Mean-CoV	Mean	μ	$\neq 0$
	CoV	$\sigma/\lvert\mu\rvert$	>0

The Standard Normal Distribution is the Normal Distribution with parameters $\mu=0$ and $\sigma=1$.


SESAM PROBAN 4.4-01
01 FEB 2002 14:47
Density function for x_normal
Normal Distribution
Density
0.0
0.1
0.2
0.3
0.4
-3
-2
-1
0
1
2
3
Variable
x_normal
Mean 0.0
Stand-Dev 1.0

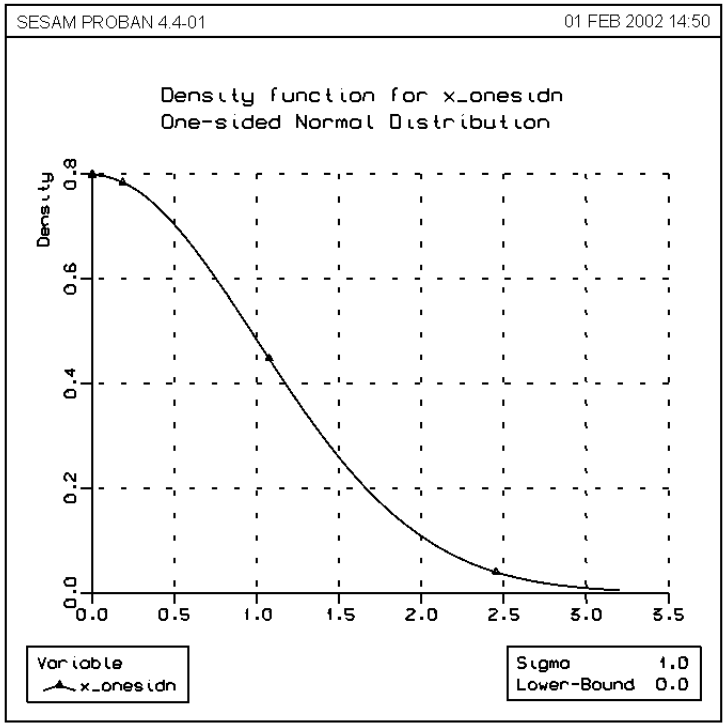
2.18 Onesided-Normal Distribution

Name	Onesi-Normal
Density	$f(x) = \frac{2}{\sqrt{2\pi}\sigma_n}\exp\left(-\frac{1}{2}\left(\frac{x-\mu_n}{\sigma_n}\right)^2\right)$
Cumulative	$F(x) = \frac{2}{\sqrt{2\pi}\sigma_n}\int_{\mu_n}^{x}\exp\left(-\frac{1}{2}\left(\frac{s-\mu_n}{\sigma_n}\right)^2\right)ds$
Restrictions	$x \geq \mu_n, \sigma > 0$
Lower bound	μ_n
Median	$m = \sigma_n\sqrt{2}\Gamma^{-1}\left(\frac{1}{2}, \frac{\sqrt{\pi}}{2}\right)^{0.5} + \mu_n$
Moments	$E((x-\mu_n)^m) = \frac{2^{\frac{m}{2}}\sigma_n^m}{\sqrt{\pi}}\Gamma\left(\frac{m+1}{2}\right)$
Mean	$\mu = \frac{\sqrt{2}}{\sqrt{\pi}}\sigma_n + \gamma_n$
Standard Deviation	$\sigma = \left(1-\frac{2}{\pi}\right)^{0.5}\sigma_n$
Skewness	$\delta = \frac{\sqrt{2}(4-\pi)}{(\pi-2)^{1.5}} = 0.9952717$
Kurtosis	$\kappa = \frac{3\pi^2-4\pi-12}{(\pi-2)^2} = 3.869177$

Parameter Set	Parameter Names	Symbol	Restriction
Mean-Low	Mean	μ	
	Lower-Bound	μ_n	$< \mu$
Sigma-Low	Sigma	σ_n	> 0
	Lower-Bound	μ_n	

The Onesided Normal Distribution is the upper half of the Normal Distribution with mean μ_n and standard deviation σ_n, and scaled to become a probability distribution function.

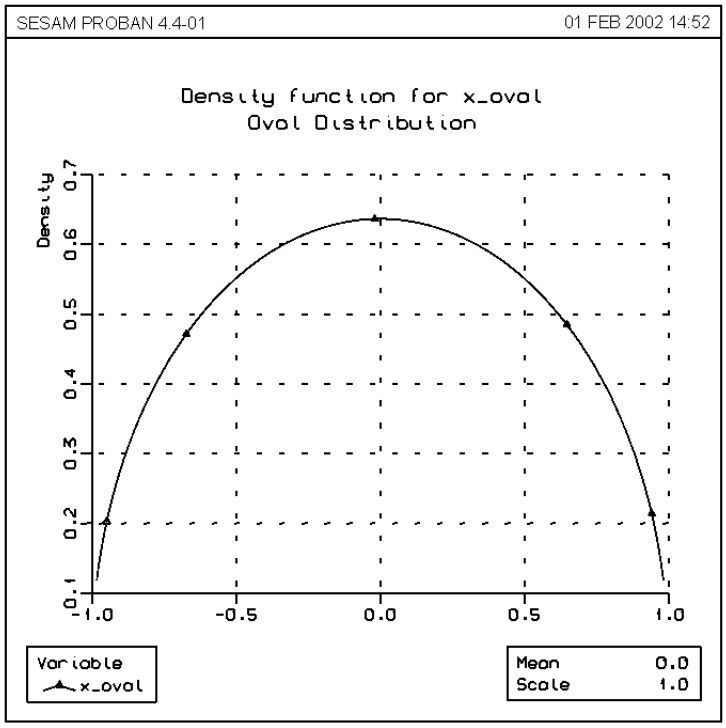
The Onesided Normal Distribution is the Generalized gamma Distribution with parameters a=0.5, b=2 and $c=\sqrt{2}\,\sigma_n$.



2.19 Oval Distribution

Name	Oval
Density	$f(x) = \frac{2}{\pi s}\left(1 - \left(\frac{x-\mu}{s}\right)^2\right)^{0.5}$
Cumulative	$F(x) = \frac{1}{\pi}\left(\frac{\pi}{2} + \frac{x-\mu}{s}\left(1 - \left(\frac{x-\mu}{s}\right)^2\right)^{0.5} + \text{Arcsin}\left(\frac{x-\mu}{s}\right)\right)$
Restrictions	$\mu - s \le x \le \mu + s,\ s > 0$
Scale	s
Median	$m = \mu$
Moments	$E((x-\mu)^n) = \frac{s^n}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+2\right)}$; if n even, else 0.
Mean	μ
Standard Deviation	$\sigma = 0.5s$
Skewness	$\delta = 0$
Kurtosis	$\kappa = 2$

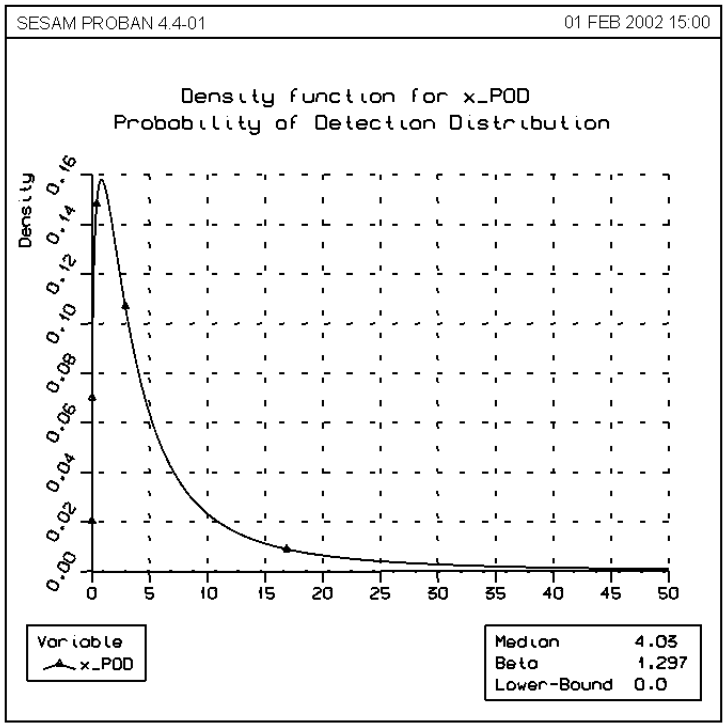
Parameter Set	Parameter Names	Symbol	Restriction
Mean-Scale	Mean	μ	
	Scale	s	> 0


SESAM PROBAN 4.4-01
01 FEB 2002 14:52
Density function for x_oval
Oval Distribution
Density
0.7
0.6
0.5
0.4
0.3
0.2
0.1
-1.0
-0.5
0.0
0.5
1.0
Variable
x_oval
Mean 0.0
Scale 1.0

2.20 Probability of Detection Distribution

Name	POD-Distrib
Density	$f(x) = \frac{(m-\gamma)^{\frac{1}{\beta}}\beta(x-\gamma)^{\beta-1}}{\left(1+\left(\frac{x-\gamma}{m-\gamma}\right)^{\beta}\right)^2}$
Cumulative	$F(x) = 1 - \frac{1}{1+\left(\frac{x-\gamma}{m-\gamma}\right)^{\beta}}$
Restrictions	$m > \gamma$, $\beta > 0$
Median	m
Moments	$E(X^n) = (m-\gamma)^n B\left(1+\frac{n}{\beta}, 1-\frac{n}{\beta}\right)$; if $\beta > n$, else ∞.
Mean	$\mu = (m-\gamma)B\left(1+\frac{1}{\beta}, 1-\frac{1}{\beta}\right)+\gamma$, if $\beta > 1$, else ∞.
Standard Deviation	$\sigma = (E(X^2)-\mu^2)^{0.5}$, if $\beta > 2$, else ∞.
Skewness	$\delta = \frac{1}{\sigma^3}(E(X^3)-3\mu E(X^2)+2\mu^3)$, if $\beta > 3$, else ∞.
Kurtosis	$\kappa = \frac{1}{\sigma^4}(E(X^4)-4\mu E(X^3)+6\mu^2 E(X^2)-3\mu^4)$, if $\beta > 4$, else ∞.

Parameter Set	Parameter Names	Symbol	Restriction
Med-Beta-Low	Median	m	
	Beta	β	> 0
	Lower-Bound	γ	$< m$

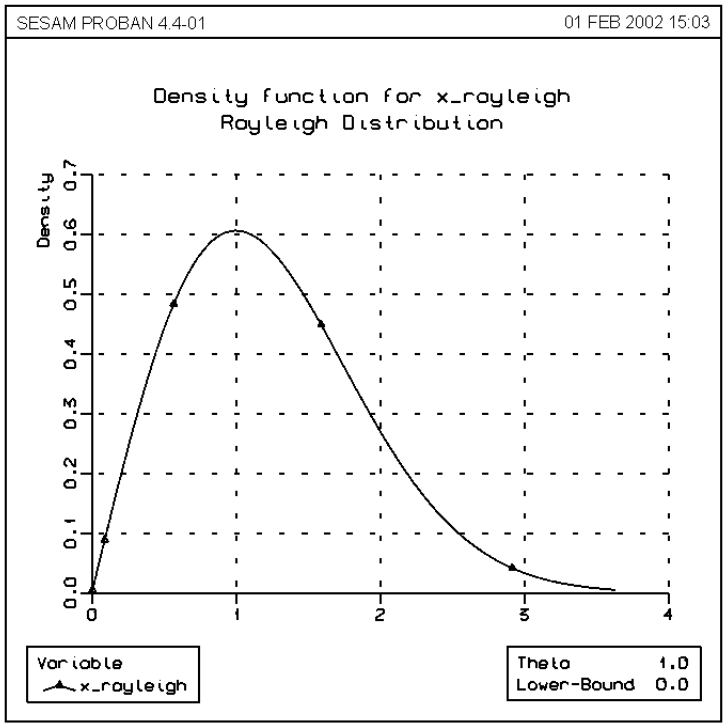

SESAM PROBAN 4.4-01
01 FEB 2002 15:00
Density function for x_POD
Probability of Detection Distribution
Density
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
0
5
10
15
20
25
30
35
40
45
50
Variable
x_POD
Median 4.03
Beta 1.297
Lower-Bound 0.0

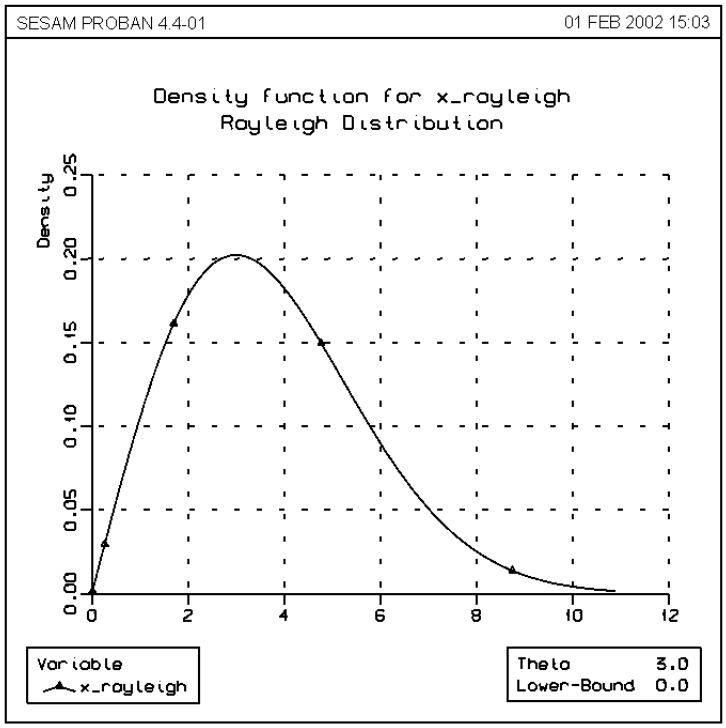
2.21 Rayleigh Distribution

Name	Rayleigh
Density	$f(x) = \frac{x-\gamma}{\theta^2}\exp\left(-\frac{1}{2}\left(\frac{x-\gamma}{\theta}\right)^2\right)$
Cumulative	$F(x) = 1 - \exp\left(-\frac{1}{2}\left(\frac{x-\gamma}{\theta}\right)^2\right)$
Restrictions	$x \geq \gamma,\ \theta > 0$
Median	$m = \theta\sqrt{2}\sqrt{-\ln(0.5)} + \gamma$
Moments	$E((X-\gamma)^n) = 2^{\frac{n}{2}}\theta^n\Gamma\left(1+\frac{n}{2}\right)$
Mean	$\mu = \frac{\sqrt{\pi}}{\sqrt{2}}\theta + \gamma$
Standard Deviation	$\sigma = \sqrt{2}\theta\left(1-\frac{\pi}{4}\right)^{0.5}$
Skewness	$\delta = \frac{2\sqrt{\pi}(2\pi-6)}{(4-\pi)^{1.5}} = 1.262221$
Kurtosis	$\kappa = \frac{32-3\pi^2}{(4-\pi)^2} = 3.245089$

Parameter Set	Parameter Names	Symbol	Restriction
Mean-Low	Mean	μ	
	Lower-Bound	γ	$< \mu$
Theta-Low	Theta	θ	> 0
	Lower-Bound	γ	

The Rayleigh Distribution is the Generalized Gamma Distribution with parameters a=1, b=2 and c=$\theta\sqrt{2}$.


SESAM PROBAN 4.4-01
01 FEB 2002 15:03
Density function for x_rayleigh
Rayleigh Distribution
Density
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0
1
2
3
4
Variable
x_rayleigh
Theta 1.0
Lower-Bound 0.0


SESAM PROBAN 4.4-01
01 FEB 2002 15:03
Density function for x_rayleigh
Rayleigh Distribution
Density
0.00
0.05
0.10
0.15
0.20
0.25
0
2
4
6
8
10
12
Variable
x_rayleigh
Theta 3.0
Lower-Bound 0.0

2.22 Splined Distribution, One Dimension

Name	Spline-1Dim
Cumulative	$F(x) = \sum_{i=1}^{n} c_i B_{i,k,t}(x)$
Restrictions	Lower Bound $< x <$ Upper Bound

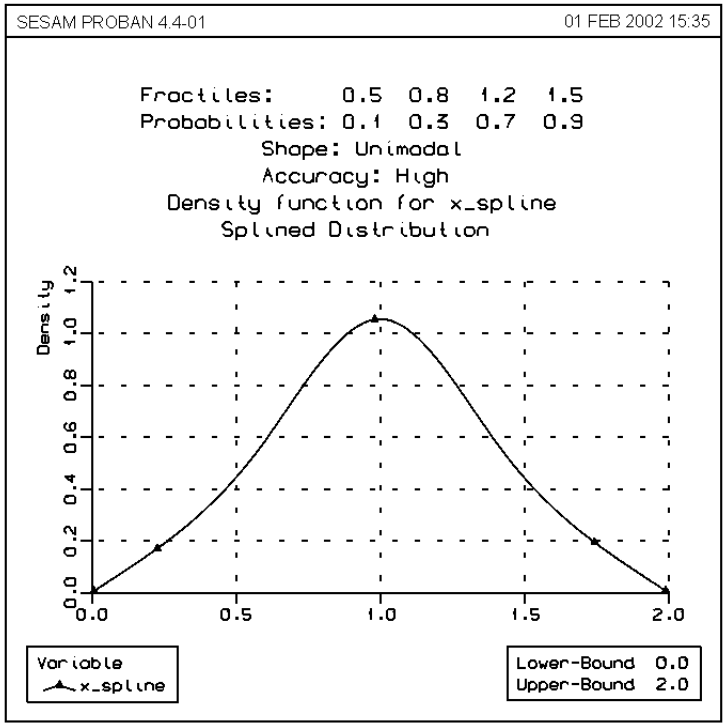
Input	Options	Description
Lower Bound		Start point for distribution
Upper Bound		End point for distribution
Weighting	Unweighted	All weights equal to 1
	Weighted	Weights are input
Accuracy		Unimodal Free Shape
	Low	n=6 n=4
	Medium	n=(m+11)/2 n=m/2+3(+1 if equal density tail values)
	High	n=m+5 n=m+2(+2 if equal density tail values)
Shape	Free	Multi modality permitted
	Unimodal	Unimodal distribution, zero density at end points
Density Tail Values	Equal	Equal at end points
	Free	Decided by algorithm
	Vanish	Forced to zero
Fractile		$m \geq 3$ distribution arguments
Cumulative probability		Corresponding probabilities
Weight		Corresponding weights if weighted

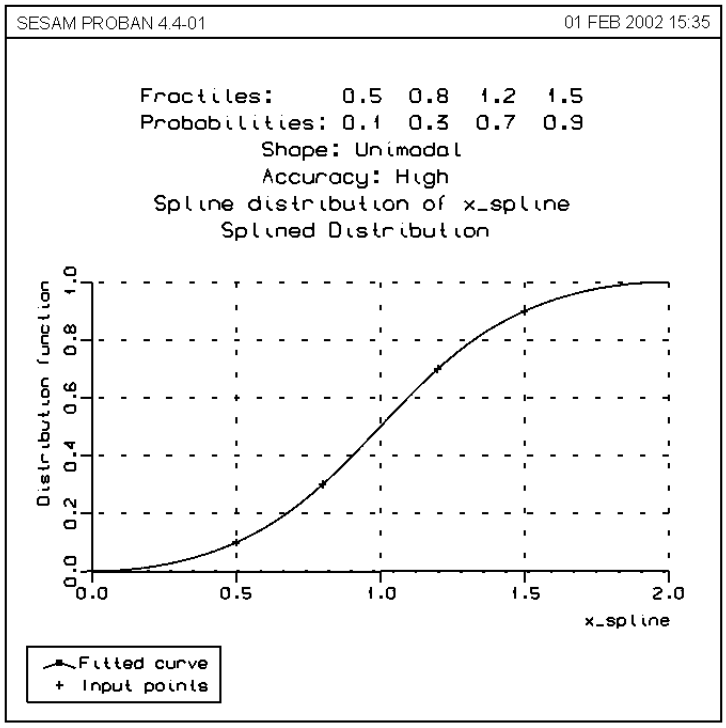
The n *B-spline* polynomials $B_{i,k,t}(x)$, i=1,2,...,n are defined on a monotone sequence of points t=t_j, j=1,2,...,n+k, the ith polynomial being positive on $t_i < x < t_{i+k}$ and zero else.

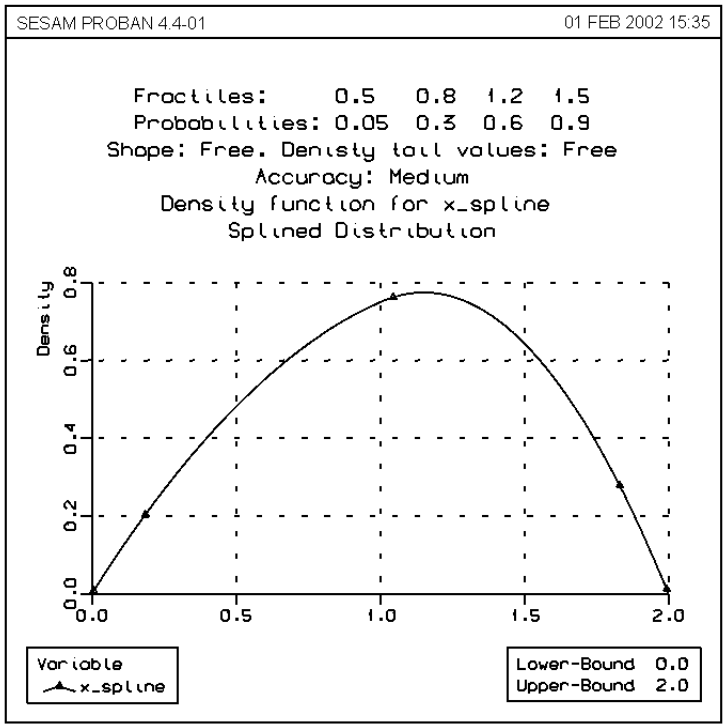
A weighted least squares method is used to calculate the vector c. Thus the following least squares sum is minimized:

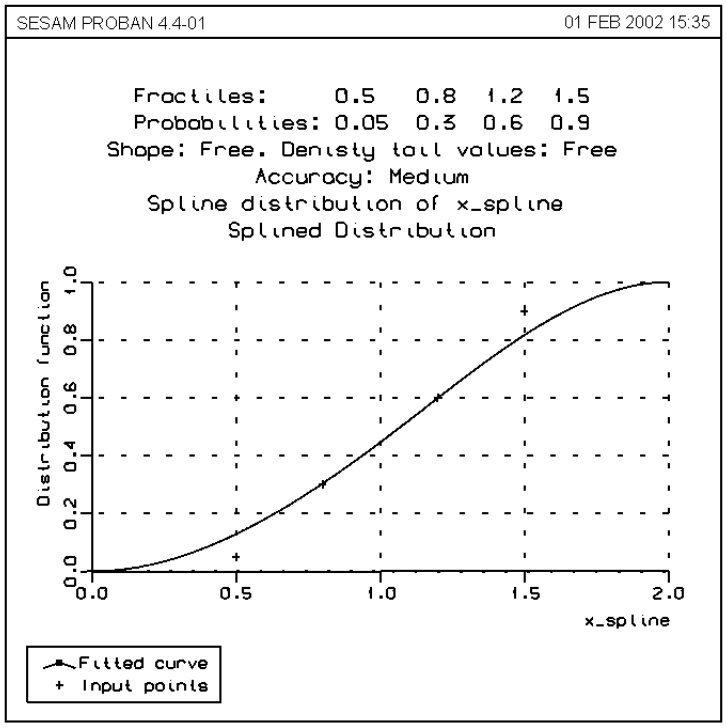
$$LSQ(\boldsymbol{c}) = \sum_{l=1}^{m} w_l (F(x_l) - y_l)^2$$

in which (x_l, y_l, w_l), l=1,2,...,*m* are the *m* fractiles, probabilities and weights respectively.




SESAM PROBAN 4.4-01
01 FEB 2002 15:35
Fractiles: 0.5 0.8 1.2 1.5
Probabilities: 0.1 0.3 0.7 0.9
Shape: Unimodal
Accuracy: High
Spline distribution of x_spline
Splined Distribution
Distribution function
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.5
1.0
1.5
2.0
x_spline
Fitted curve
Input points


SESAM PROBAN 4.4-01
01 FEB 2002 15:35
Fractiles: 0.5 0.8 1.2 1.5
Probabilities: 0.05 0.3 0.6 0.9
Shape: Free. Density tail values: Free
Accuracy: Medium
Density function for x_spline
Splined Distribution
Density
0.0
0.2
0.4
0.6
0.8
0.0
0.5
1.0
1.5
2.0
Variable
x_spline
Lower-Bound 0.0
Upper-Bound 2.0


SESAM PROBAN 4.4-01
01 FEB 2002 15:35
Fractiles: 0.5 0.8 1.2 1.5
Probabilities: 0.05 0.3 0.6 0.9
Shape: Free. Density tail values: Free
Accuracy: Medium
Spline distribution of x_spline
Splined Distribution
Distribution function
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.5
1.0
1.5
2.0
x_spline
Fitted curve
Input points

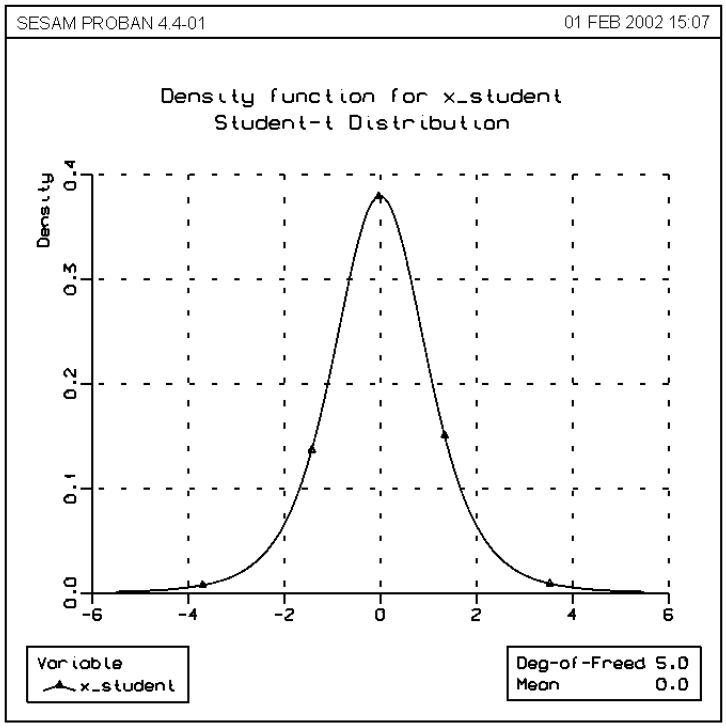
2.23 Student-t Distribution

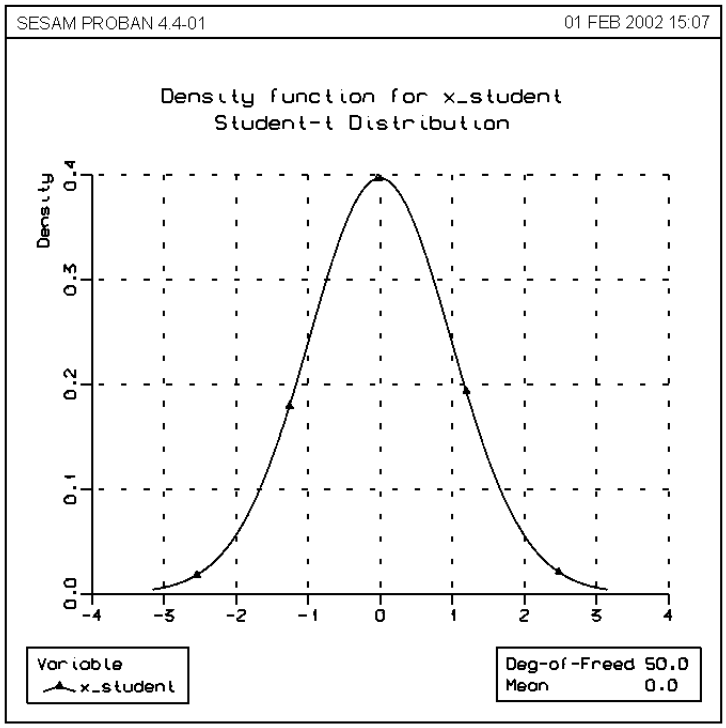
Name	Student-t
Density	$f(x) = \frac{1}{\sqrt{\nu}B\left(\frac{1}{2}, \frac{\nu}{2}\right)}\left(1 + \frac{(x-\mu)^2}{\nu}\right)^{-\frac{\nu+1}{2}}$
Cumulative	$F(x) = \frac{1}{\sqrt{\nu}B\left(\frac{1}{2}, \frac{\nu}{2}\right)}\int_{-\infty}^{x}\left(1 + \frac{(s-\mu)^2}{\nu}\right)^{-\frac{\nu+1}{2}} ds$
Restrictions	$\nu \geq 2$
Degrees of Freedom	ν
Median	$m = \mu$
Moments	$E((X-\gamma)^n) = \frac{\nu^{\frac{n}{2}}}{\sqrt{\pi}}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{\nu-n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}$, n even, 0 else.
Mean	μ
Standard Deviation	$\sigma = \left(1 - \frac{2}{\nu}\right)^{-0.5}$
Skewness	$\delta = 0$
Kurtosis	$\kappa = \frac{3(\nu-2)}{(\nu-4)}$, $\nu > 4$

Parameter Set	Parameter Names	Symbol	Restriction
DoF-Mean	Deg-of-Freed	ν	≥ 2
	Mean	μ	

The Student-t Distribution is the ratio of a Standard Normal distributed random variable to the square root of a χ^2 distributed random variable divided by its number of degrees of freedom.

A Student-t distributed random variable may be replaced by a Normal distributed random variable with mean = μ and standard deviation $\sigma = 1$ if ν is large, say $\nu \geq 100$.

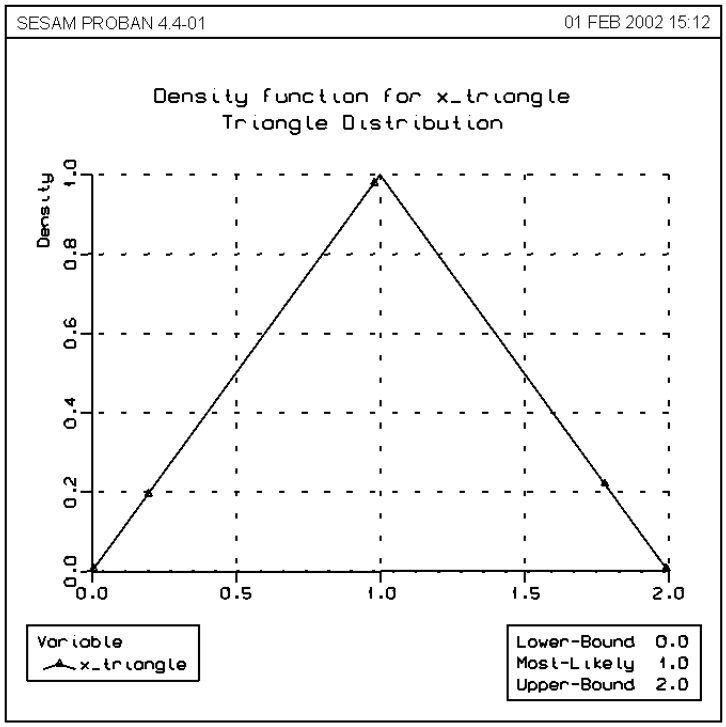

SESAM PROBAN 4.4-01
01 FEB 2002 15:07
Density function for x_student
Student-t Distribution
Density
0.0
0.1
0.2
0.3
0.4
-6
-4
-2
0
2
4
6
Variable
x_student
Deg-of-Freed 5.0
Mean 0.0

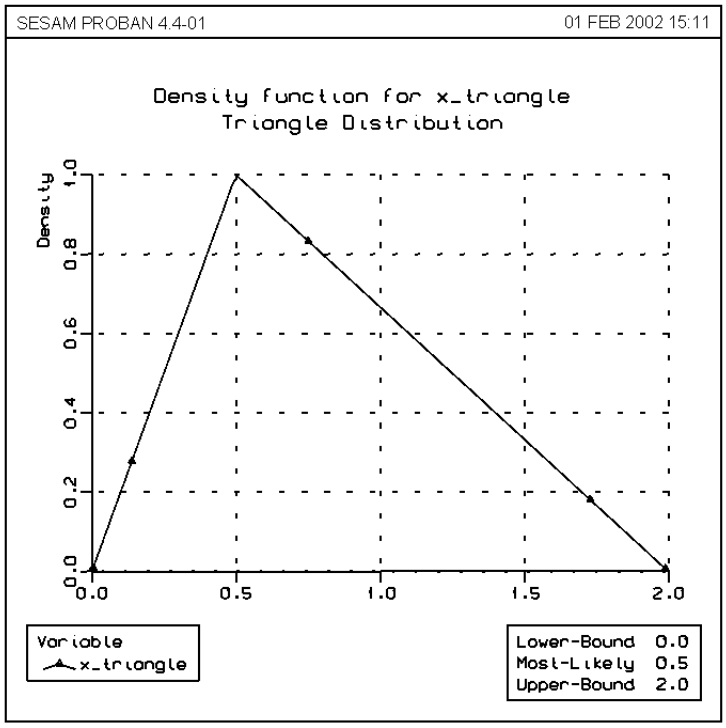

SESAM PROBAN 4.4-01
01 FEB 2002 15:07
Density function for x_student
Student-t Distribution
Density
0.0
0.1
0.2
0.3
0.4
-4
-3
-2
-1
0
1
2
3
4
Variable
x_student
Deg-of-Freed 50.0
Mean 0.0

2.24 Triangle Distribution

Name	Triangle
Density	$f(x) = \begin{cases} 2\dfrac{x-a}{(m-a)(b-a)}, & a \le x \le m \\ 2\dfrac{b-x}{(b-m)(b-a)}, & m < x \le b \end{cases}$
Cumulative	$F(x) = \begin{cases} 2\dfrac{(x-a)^2}{(m-a)(b-a)}, & a \le x \le m \\ \left(1 - \dfrac{(b-x)^2}{(b-m)(b-a)}\right), & m < x \le b \end{cases}$
Restrictions	$a \le x \le b, a \le m \le b, \dfrac{2a+b}{3} \le \mu \le \dfrac{a+2b}{3}$
Most likely value	m
Lower Bound	a
Upper Bound	b
Median	$m = \begin{cases} a + \sqrt{0.5 + (m-a)(b-a)}, F(m) \le 0.5 \\ b - \sqrt{0.5 + (b-m)(b-a)}, F(m) > 0.5 \end{cases}$
Moments	$E((X-\gamma)^n) = 2\dfrac{(b-m)^{n+1} + (-1)^n (m-a)^{n+1}}{(b-a)(n+1)(n+2)}$
Mean	$\mu = \dfrac{a+b+m}{3}$
Standard Deviation	$\sigma = \dfrac{\sqrt{a^2+b^2+m^2-ab-am-bm}}{\sqrt{18}}$
Skewness	$\delta = \dfrac{\sqrt{2}}{5}\dfrac{2a^2+2b^2-5ab+ma+mb-m^2}{(a^2+b^2+m^2-ab-am-bm)^{1.5}}(a+b-2m)$

Parameter Set	Parameter Names	Symbol	Restriction
Low-MostL-Up	Lower-Bound	a	$\leq m$
	Most-Likely	m	
	Upper-Bound	b	$\geq m$
Low-Mean-Up	Lower-Bound	a	$< \mu$
	Mean	μ	$(2a+b)/3 <= \mu <= (a+2b)/3$
	Upper-Bound	b	$> \mu$



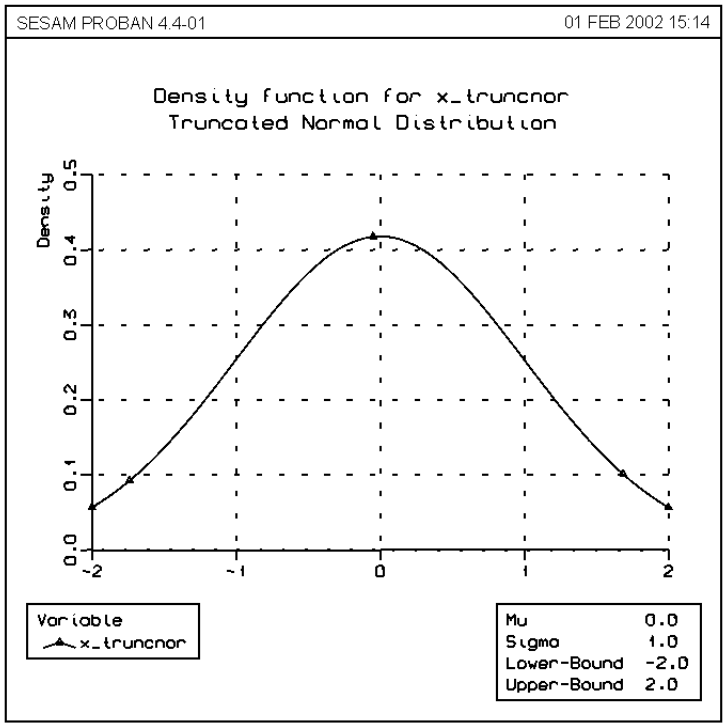

SESAM PROBAN 4.4-01
01 FEB 2002 15:11
Density function for x_triangle
Triangle Distribution
Density
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.5
1.0
1.5
2.0
Variable
x_triangle
Lower-Bound 0.0
Most-Likely 0.5
Upper-Bound 2.0

2.25 Truncated Normal Distribution

Name	Trunc Normal
Density	$f(x) = \dfrac{\frac{1}{\sigma_n}\phi\left(\frac{x-\mu_n}{\sigma_n}\right)}{\Phi\left(\frac{b-\mu_n}{\sigma_n}\right) - \Phi\left(\frac{a-\mu_n}{\sigma_n}\right)}$
Cumulative	$F(x) = \dfrac{\Phi\left(\frac{x-\mu_n}{\sigma_n}\right) - \Phi\left(\frac{a-\mu_n}{\sigma_n}\right)}{\Phi\left(\frac{b-\mu_n}{\sigma_n}\right) - \Phi\left(\frac{a-\mu_n}{\sigma_n}\right)}$
Restrictions	$a \le x \le b, \sigma_n > 0$
Definitions	$s_a = sign(a-\mu_n); s_b = sign(b-\mu_n); A = \dfrac{a-\mu_n}{\sigma_n}; B = \dfrac{b-\mu_n}{\sigma_n}$
Moments	$E\left(\left(\dfrac{X-\mu_n}{\sigma_n\sqrt{2}}\right)^m\right) = \dfrac{s_b^{m+1}\gamma\left(\frac{m+1}{2}, \frac{B^2}{2}\right) - s_a^{m+1}\gamma\left(\frac{m+1}{2}, \frac{A^2}{2}\right)}{2\sqrt{\pi}(\Phi(B) - \Phi(A))}$
Mean	$\mu = \sigma_n \dfrac{\phi(A) - \phi(B)}{\Phi(B) - \Phi(A)} + \mu_n$
Standard Deviation	$\sigma = \sigma_n \left\{ 1 + \dfrac{A\phi(A) - B\phi(B)}{\Phi(B) - \Phi(A)} - \left(\dfrac{\phi(A) - \phi(B)}{\Phi(B) - \Phi(A)}\right)^2 \right\}^{0.5}$

Parameter Set	Parameter Names	Symbol	Restriction
Mu-Sigma-Lim	Mu	μ_n	
	Sigma	σ_n	> 0
	Lower-Bound	a	
	Upper-Bound	b	> a

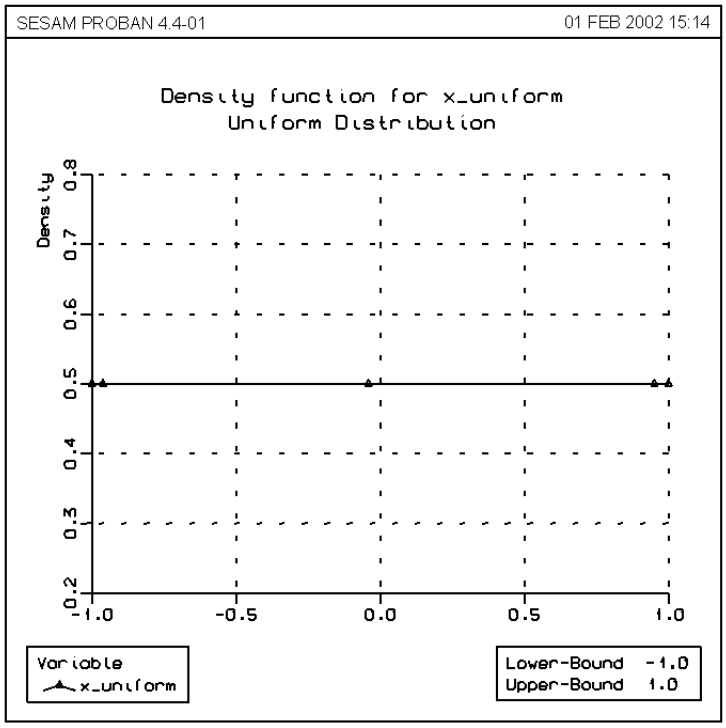
Mu-CoV-Lim	Mu	μ_n	$\neq 0$
	Coef-of-Var	$\sigma_n/\lvert\mu_n\rvert$	> 0
	Lower-Bound	a	
	Upper-Bound	b	$> a$



2.26 Uniform Distribution

Name	Uniform
Density	$f(x) = \frac{1}{b-a}$
Cumulative	$F(x) = \frac{x-a}{b-a}$
Restrictions	$a \le x \le b$
Lower Bound	a
Upper Bound	b
Median	$\frac{a+b}{2}$
Moments	$E((X-\mu)^n) = \frac{1}{2^n}\frac{1}{n+1}(b-a)^n$, n even, 0 else.
Mean	$\mu = \frac{a+b}{2}$
Standard Deviation	$\sigma = \frac{b-a}{2\sqrt{3}}$
Skewness	$\delta = 0$
Kurtosis	$\kappa = 1.8$

Parameter Set	Parameter Names	Symbol	Restriction
Limits	Lower-Bound	a	
	Upper-Bound	b	$> a$
Mean-Low	Mean	μ	
	Lower-Bound	a	$< \mu$
Mean-StD	Mean	μ	
	Stand-Dev	σ	> 0
Mean-CoV	Mean	μ	$\neq 0$
	Coef-of-Var	$\sigma/\lvert\mu\rvert$	> 0

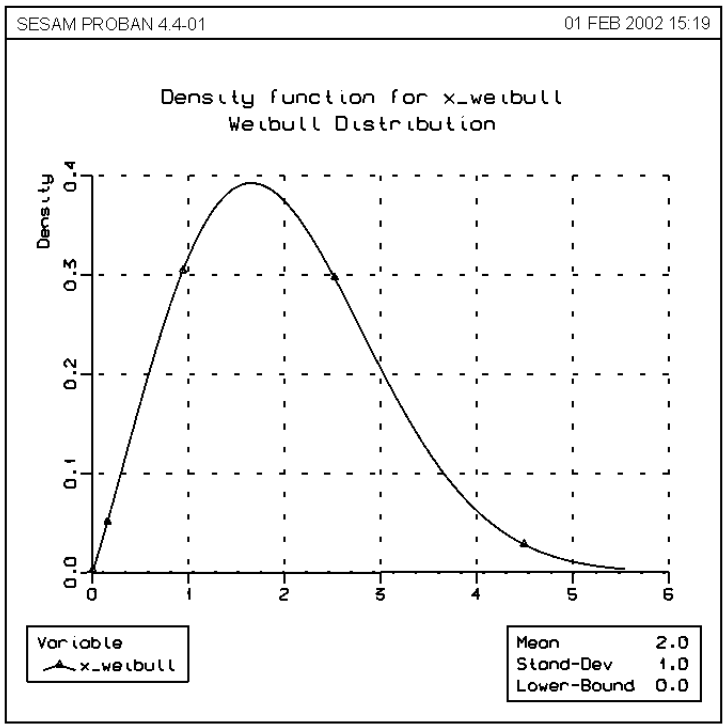

SESAM PROBAN 4.4-01
01 FEB 2002 15:14
Density function for x_uniform
Uniform Distribution
Density
0.8
0.7
0.6
0.5
0.4
0.3
0.2
-1.0
-0.5
0.0
0.5
1.0
Variable
x_uniform
Lower-Bound -1.0
Upper-Bound 1.0

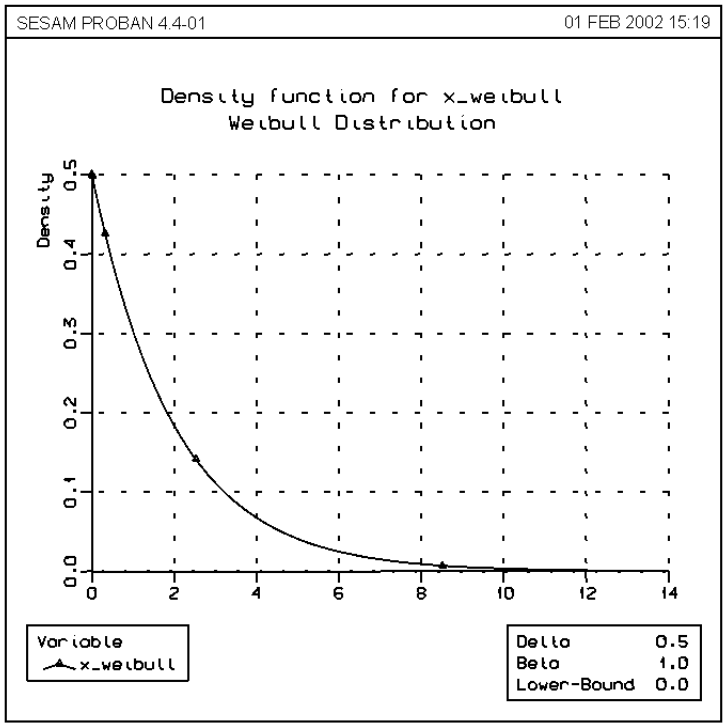
2.27 Weibull Distribution

Name	Weibull
Density	$f(x) = \delta_w \beta (x-\gamma)^{\beta-1} \exp(-\delta_w (x-\gamma)^{\beta})$
Cumulative	$F(x) = 1 - \exp(-\delta_w (x-\gamma)^{\beta})$
Restrictions	$x \geq \gamma,\ \delta_w > 0,\ \beta > 0$
Lower Bound	γ
Alpha	$\alpha = \delta_w^{-\frac{1}{\beta}}$
Median	$\left(\frac{-\ln(0.5)}{\delta_w}\right)^{\frac{1}{\beta}} + \gamma$
Moments	$E((X-\gamma)^n) = \delta_w^{-\frac{n}{\beta}} \Gamma\left(1+\frac{n}{\beta}\right) = \alpha^n \Gamma\left(1+\frac{n}{\beta}\right)$
Mean	$\mu = \alpha \Gamma\left(1+\frac{1}{\beta}\right) + \gamma$
Standard Deviation	$\sigma = \alpha\left(\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma\left(1+\frac{1}{\beta}\right)^2\right)^{0.5}$
Skewness	$\delta = \frac{\Gamma\left(1+\frac{3}{\beta}\right) - 3\Gamma\left(1+\frac{1}{\beta}\right)\Gamma\left(1+\frac{2}{\beta}\right) + 2\Gamma\left(1+\frac{1}{\beta}\right)^3}{\left(\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma\left(1+\frac{1}{\beta}\right)^2\right)^{1.5}}$
Kurtosis	$s = \Gamma\left(1+\frac{4}{\beta}\right) - 4\Gamma\left(1+\frac{1}{\beta}\right)\Gamma\left(1+\frac{3}{\beta}\right)$ $t = 6\Gamma\left(1+\frac{1}{\beta}\right)^2 \Gamma\left(1+\frac{2}{\beta}\right) - 3\Gamma\left(1+\frac{1}{\beta}\right)^4$ $\kappa = \frac{s+t}{\left(\Gamma\left(1+\frac{2}{\beta}\right) - \Gamma\left(1+\frac{1}{\beta}\right)^2\right)^2}$

Parameter Set	Parameter Names	Symbol	Restriction
Mean-StD-Low	Mean	μ	
	Stand-Dev	σ	> 0
	Lower-Bound	γ	$< \mu$
Mean-CoV-Low	Mean	μ	$\neq 0$
	Coef-of-Var	$\sigma/\lvert\mu\rvert$	> 0
	Lower-Bound	γ	$< \mu$
Delt-Bet-Low	Delta	δ_w	> 0
	Beta	β	> 0
	Lower-Bound	γ	
Alp-Bet-Low	Alpha	α	> 0
	Beta	β	> 0
	Lower-Bound	γ	

The Weibull Distribution is the Generalized Gamma Distribution with parameters a=1, b=β, and c= α.




SESAM PROBAN 4.4-01
01 FEB 2002 15:19
Density function for x_weibull
Weibull Distribution
Density
0.0
0.1
0.2
0.3
0.4
0.5
0
2
4
6
8
10
12
14
Variable
x_weibull
Delta 0.5
Beta 1.0
Lower-Bound 0.0

3 DISTRIBUTION LIBRARY - DISCRETE DISTRIBUTIONS

3.1 Introduction

The Proban Distribution Library -Discrete Distributions is a built-in collection of discrete distributions. The available ones are given in Table .

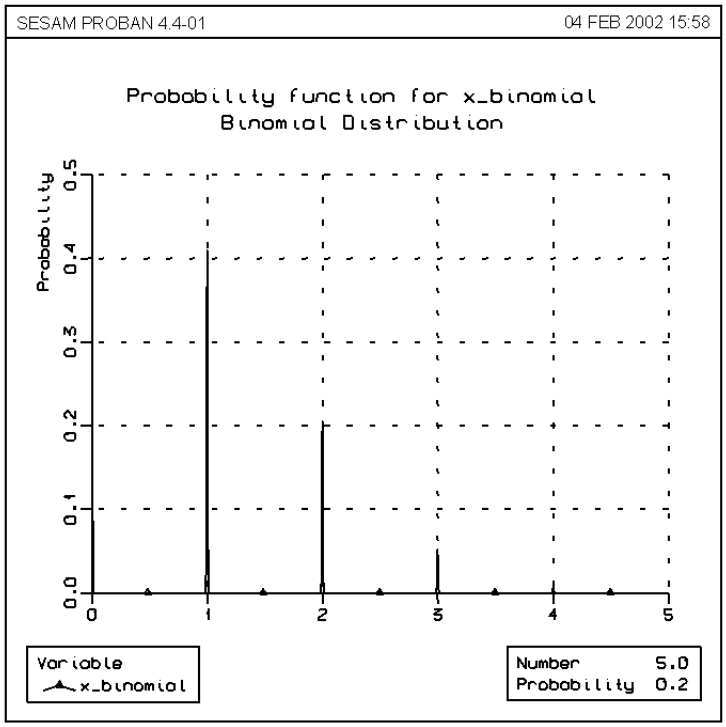
.

Binomial	Binomial Distribution
Poisson	Poisson Distribution

3.2 Binomial Distribution

Name	Binomial
Pointwise Probability	$p(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$
Cumulative	$F(x) = \sum_{z \le x} p(z)$
Restrictions	$x \in (0, 1, 2, \ldots, n); 0 \le p \le 1$
Number of Observations	n
Probability	p
Mean	$\mu = np$
Standard Deviation	$\sigma = \sqrt{(np(1-p))}$

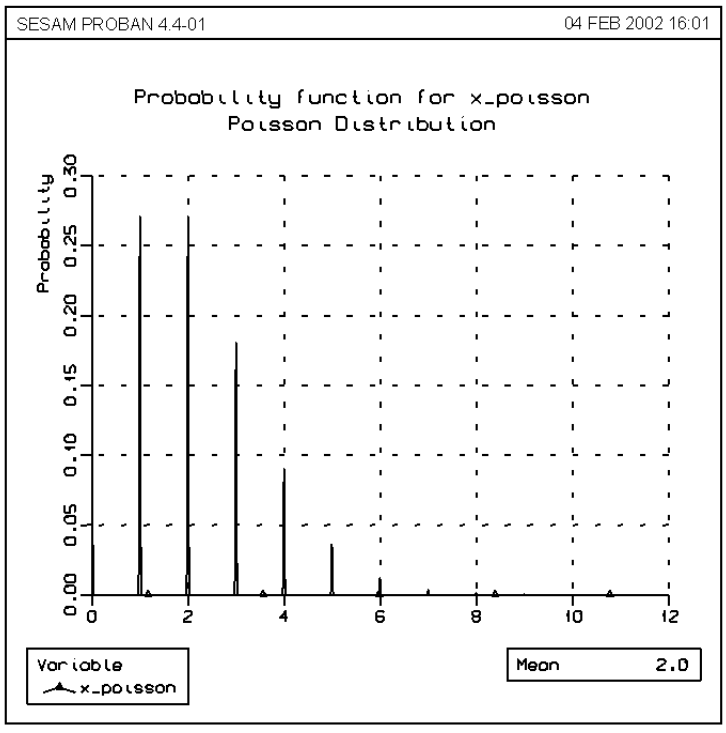
Parameter Set	Parameter Names	Symbol	Restriction
N-Probab	Number	n	> 0
	Probability	p	$0 \le p \le 1$


SESAM PROBAN 4.4-01
04 FEB 2002 15:58
Probability function for x_binomial
Binomial Distribution
Probability
0.0
0.1
0.2
0.3
0.4
0.5
0
1
2
3
4
5
Variable
x_binomial
Number 5.0
Probability 0.2

3.3 Poisson Distribution

Name	Poisson
Pointwise Probability	$p(x) = \frac{\mu^x \exp(-\mu)}{x!}$
Cumulative	$F(x) = \sum_{z \le x} p(z)$
Restrictions	$x \in (0, 1, 2, \ldots, n); \mu \ge 0$
Mean	μ
Standard Deviation	$\sigma = \sqrt{\mu}$

Parameter Set	Parameter Names	Symbol	Restriction
Mean	Mean	μ	> 0



4 OTHER DISTRIBUTIONS

4.1 Generated Distribution

4.1.1 General

The Generated Distribution permits a function of random variables as a distribution type.

Let Z be a random variable and $\boldsymbol{W}$ the variables of the model for Z. Create a variable X as a Generated Distribution of Z to assign the distribution of Z to X.

Conditioning of X on a subset Y of the variables $\boldsymbol{W}$, yields $X(\boldsymbol{y};\alpha)$ distributed as $Z(\boldsymbol{y})$. The values of the conditioning variables $\boldsymbol{y}$, together with the numerical parameters, α, in the model for $Z(\boldsymbol{y})$ become parameters for the distribution of X.

The random variables $X_1(y_1;\alpha_1)$ and $X_2(y_2;\alpha_2)$ generated from the same random variable Z or different random variables Z_1 and Z_2 are treated as mutually independent (zero correlation) unless a correlation is specified for $X_1(y_1;\alpha_1)$ and $X_2(y_2;\alpha_2)$ by use of the Nataf correlation model.

Create X as the Generated Distribution of Z, Z being distributed as W_1+W_2, where W_i, i=1,2, is distributed as Normal (μ_i,σ_i^2). The parameters μ_i and σ_i are constants.

This allows three distribution models for Z.

1 When W_1 and W_2 are not conditioned on then $X(\mu_1,\mu_2,\sigma_1,\sigma_2)$ is distributed as Normal$(\mu_1+\mu_2,(\sigma_1^2+\sigma_2^2))$.

2 When W_1 but not W_2 is conditioned on then X(w_1,μ_2,σ_2) is distributed as Normal$(w_1+\mu_2,\sigma_2^2)$.

3 When W_2 but not W_1 is conditioned on then X(w_2,μ_1,σ_1) is distributed as Normal($\mu_1 + w_2$,σ_1^2).

The present example gives three possible Normal distributions for $X(\boldsymbol{y};\alpha)$ and $X(\boldsymbol{y};\alpha)$ may therefore (perhaps preferably) be modelled as such. However, in general the resulting distribution for $X(\boldsymbol{y};\alpha)$ does not correspond to a standard distribution type. The Generated Distribution therefore yields a forceful extension to the random variable modelling in Proban.

The distribution of $X(\boldsymbol{y})$ is calculated using FORM - First Order Reliability Method - on $Z(\boldsymbol{y}) \leq x$. The resulting distribution of $X(\boldsymbol{y})$ is therefore a FORM approximation to the true distribution of $X(\boldsymbol{y})$.

4.1.2 Proban: Create X as Generated Distribution of Z(y).

The variable X is created by use of the following commands.

```
CREATE VARIABLE X 'Generated by Z(y)' GENERATED Z
ASSIGN CONDITIONING X ( ONLY W1 W5 W7 )
```

Two other commands are useful:

```
CHANGE VARIABLE X ...
DEFINE ANALYSIS OPTION GENERATED DISTRIBUTION
```

4.2 Extreme Value Distributions

4.2.1 General

When X is a univariate continuous random variable, i.e., a generated distribution or a continuous univariate random variable from the built-in distributions library, then X_n, the maximum or minimum of n independent realizations of X can be modelled.

4.2.2 Maximum of n Independent Realizations of X

The distribution of X_n becomes:

$$F_{X_n}(x_n) = F_X(x)^n \tag{4.1}$$

in which $F_X(x)$ is the cumulative distribution of X.

4.2.3 Minimum of n Independent Realizations of X

The distribution of X_n becomes:

$$F_{X_n}(x_n) = 1 - (1 - F_X(x))^n \tag{4.2}$$

in which $F_X(x)$ is the cumulative distribution of X.

4.2.4 Some Notes

If a parameter τ of the distribution of X is modelled as a random variable, then the current realization of τ becomes a distribution parameter for X_n.

$$F_{X_n}(x_n;\tau) = F_X(x;\tau)^n \tag{4.3}$$

and

$$F_{X_n}(x_n;\tau) = 1 - (1 - F_X(x;\tau))^n \tag{4.4}$$

For a Generated Distribution the values of the variables conditioned on together with numerical values in the model for X serves as distribution parameters.

When a max/min extreme value distribution is modelled, the variable named X takes on the distribution of X_n.

A random variable which takes on a max/min extreme value distribution may be correlated with other continuously distributed random variables. The correlation is then with X_n and not with X.

4.2.5 Proban: Create X_n as Maximum of n Independent Realizations of X

The variable X_{20} is created by use of the following command.

```
ASSIGN EXTREME-VALUE X MAX-OF-N 20
```

4.3 Correlation

4.3.1 Nataf Model

The Nataf model permits correlation also between non-Normal random variables. For example, if X_1 is Weibull distributed and X_2 Lognormal distributed, then a correlation ρ_{X_1,X_2} between them is permitted.

The Nataf distribution model for X_1 and X_2 with correlation ρ_{X_1,X_2} is obtained by mapping of X_1 and X_2 onto the Standard Normal random variables V_1 and V_2 by

$$v_i = \Phi^{-1}(F_{x_i}(x_i)); i = 1, 2 \tag{4.5}$$

and thereafter by defining their joint distribution to be the Bivariate Normal distribution:

$$f_{V_1,V_2}(v_1,v_2) = \frac{1}{2\pi\sqrt{1-\rho_{V_1,V_2}^2}} \exp\left(-\frac{v_1^2 + v_2^2 - 2v_1v_2\rho_{V_1,V_2}}{2(1-\rho_{V_1,V_2}^2)}\right) \tag{4.6}$$

There is a one-one relation between the correlations ρ_{X_1,X_2} and ρ_{V_1,V_2} so that the correlation coefficient may optionally be entered as

Basic, i.e., ρ_{X_1,X_2}

or as

Normal, i.e., ρ_{V_1,V_2}.

The variables X_1 and X_2 may both be correlated with other variables. The only restriction is that the resulting correlation matrix, $\rho_{\mathbf{V}}$, for the normalized variables must be positive definite, i.e., the correlations must be valid and not produce linear dependency between variables.

If X_1 and X_2 are jointly Normal, then the Nataf model yields the Bivariate Normal distribution. (In Proban: The Multivariate Normal distribution of dimension 2.) If X_1 and X_2 are jointly Lognormal, the Nataf model yields the usual definition of the Multilognormal distribution.

If X_1 or X_2 is a Generated Distribution, then the correlation input is restricted to ρ_{V_1,V_2}.

The Nataf model is the natural generalisation of the Multivariate Normal distribution to non-normal random variables. The correlation matrix, $\rho_{\mathbf{V}}$, measures linear dependency between the corresponding normalised variables which, in general, implies non-linear dependency between the basic variables.

If a multivariate distribution model is not precisely known, then experience shows that the Nataf model may still be useful.

4.3.2 References

Nataf,A., “Determination des Distribution dont les Marges sont Donnees,” *Comptes Rendus de l’Academie des Sciences*, Paris1962, Vol 225, pp. 42-43.

Winterstein, S.R., De, R.S., and Bjerager, P., “Correlated Non-Gaussian Models in Offshore Structural reliability,” *Proceedings, 5th International Conference on Structural Safety and Reliability*, San Francisco, California, August 1989, pp 239-242.

Liu, P.L., Der Kiureghian, A., “Multivariate Distribution Models with Prescribed Marginals and Covariances,” *Probabilistic Engineering Mechanics*, 1986, Vol 1, No 2, pp. 105-112.

4.4 Conditional Distributions

4.4.1 General

A random variable is allowed as a model for a parameter of another random variable. The only restriction is that a variable may not (in)directly reference itself.

As an example, let the variable *X* be normally distributed with mean being the sum *S* of *Y*, *W* and *Z*, and standard deviation 0.3. Create the variables *Y*, *W* and *Z*, and a new variable *S*=Sum(*Y*,*W*,*Z*). Create a variable *X* as normally distributed with mean *S* and standard deviation 0.3. The distribution of *X* is now conditioned on the current value of *S*.

The theory which permits this modelling is the Rosenblatt Transformation. This transformation maps the random variables (X_1,X_2,...,X_n) onto a set of independent standard normal random variables (U_1,U_2,...,U_n). The basic form of the transformation is:

$$\begin{aligned}
\Phi(u_1) &= F_{X_1}(x_1) \\
\Phi(u_2) &= F_{X_2}(x_2 | x_1) \\
\ldots &= \ldots \\
\Phi(u_n) &= F_{X_n}(x_n | x_1, x_2, \ldots, x_{n-1})
\end{aligned}$$

Note that many multivariate distributions are actually standard one-dimensional distributions with random parameters.

Rosenblatt,N., “Remarks on a Multivariate Transformation,” *Annals of Mathematical Statistics,* Vol 23, pp. 470-472.

4.5 Continuous Time Dependent Stochastic Process

4.5.1 General

A continuous time dependent stochastic process $X(t)$ can be seen as a particle that moves in time. An infinite number of replications of the process yields statistical uncertainty for both the position of the particle and the slope of its instant movement and also for its correlation with other particles.

A time dependent continuous process has a starting time T_S, and a duration D, measured from the starting time. The Rice formulation, (Rice 1944), in which the statistics of the process is represented in terms of the process variable $X(t)$, its time derivative $\dot{X}(t)$, and their joint statistics is employed. The actual calculation method is the parallel system sensitivity factor method based on Madsens Formula, (Hagen and Tvedt, 1991).

The calculation method implemented requires the mean of $\dot{X}(t)$ to be zero, i.e., drift cannot be modelled through the time derivative. As shown below, a model for a time dependent stochastic process can always be formulated to comply with this restriction.

The variables $X(t)$ and $\dot{X}(t)$ have continuous distributions. The continuous distribution models in the Proban Distributions Library, the Generated distribution, the Max and Min Extreme value distributions, conditional distributions and correlated distributions are all permitted.

Let $\dot{X}(t)$ have mean $\dot{\mu}(t)$ and let failure be described by

$$G(X(t)) < 0 \tag{4.7}$$

Then by using

$$Y(t) = X(t) - \gamma(t) \tag{4.8}$$

there is

$$G(Y(t) + \gamma(t)) < 0 \tag{4.9}$$

Since

$$\dot{Y}(t) = \dot{X}(t) - \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) \tag{4.10}$$

there is

$$\gamma(t) = \int_{T_S}^{t} \dot{\mu}(s)ds \tag{4.11}$$

and $\dot{Y}(t)$ has zero mean. As an example, if the expectation $\dot{\mu}(t)$ is a constant C, then $\gamma(t) = C(t - T_S)$

Rice, S.O., "Mathematical Analysis of Random Noise", *Bell System Technical Journal*, pp. 46-156. Reprinted in *Selected Papers in Noise and Stochastic Processes*, ed. N. Wax, Dover, New York, 1954.

Hagen, Ø., and Tvedt, L., "Vector Process Out-Crossing as Parallel System Sensitivity Measure", *Journal of Engineering Mechanics*, Vol 117, No. 10, October 1991.

4.5.2 Create a Continuous Time Dependent Process

Create one variable, X, to model the position of $X(t)$ and a variable XDOT to model $\dot{X}(t)$. Assign XDOT as the time derivative of X.

```
ASSIGN CONTINUOUS-PROCESS X XDOT
```

Explicit time dependency is modelled through the variable TIME.

4.6 Distributions Defined by User (UNIX)

4.6.1 User Defined Distributions

To add a user defined distribution to Proban requires that the distribution is programmed, and then linked into Proban.

Use the following sequence to add a user defined distribution:

1 Select a three character routine prefix for the distribution. This prefix should begin the name of each routine programmed with the distribution. For illustration of the process, assume that the chosen prefix is XXX. These prefixes cannot be used:

ATR BET BPM CDI CH2 CGR CHK CIQ COP CPM DDI DES DFU DIM DIS EXP EXT FOX FU FX GAM GGM GUM HTM ICO IDI IIQ ING INI IPM IQ IQC LNM LOH LSC LSD LSI MNR MOM MSG MXW NAM NMS NPM NRM NUM ONE OP OVA PAR PM PMI PMN PTZ RAY SP1 SP2 STN STU TAC TOC TPA TRA TRI TRU TST UAT UNI USR VTZ WBL ZTV ZVP

2 The distribution, input sequence and parameter(s) must be installed by modifying the routine USRINI. During start-up, Proban calls USRINI to install any user defined distributions. The delivered version of USRINI does not install any user defined distributions.

The distribution is allowed to have one input sequence. The input sequences that are in use in Proban already may be reused, or a new input sequence may be installed. The same applies to the parameters in the input sequence. If an existing input sequence or parameter is used, all restrictions that apply to the input sequence and parameters will also be in effect for the new distributions.

Other details about the installation are described in USRINI itself. The location of USRINI is described in the installation guide. At the same place there is an example routine: USRINI.TST showing how the TST distribution is implemented.

3 Program the DDI routine for the distribution (e.g. named XXXDDI). This routine calculates the density function, distribution function and complementary distribution function from a fractile in the distribution. Proban is delivered with an example, called TSTDDI, that should be used as a template for the routine. The location of this routine is specified in the installation guide.

Proban requires good accuracy in the tail of the distribution, and may call the DDI routine with extreme tail values. Please be aware of this, and take special note of the possibility of an overflow (e.g. in the exp function) if a tail value is extreme.

4 The DDI routine is activated through the routine USRDDI. USRDDI must be modified by inserting a call to the DDI routine for the distribution. See the documentation in USRDDI itself for further clarification. Proban is delivered with a USRDDI routine that does not call any user defined distributions. The location of USRDDI is described in the installation guide. At the same place there is an example routine: USRDDI.TST showing how the TST distribution is implemented.

5 Proban is delivered with an object library, called USER. The location of the library is described in the installation guide. This library contains the user defined distributions (it is delivered with only USRINI and USRDDI). Take a copy of this library. Then compile USRINI, USRDDI and the distribution DDI

routine (e.g. XXXDDI) (and possibly other routines that are needed by the new DDI routine) and place the object codes in the USER library.

6 Link the USER library into Proban using the link command file or makefile delivered with Proban The procedure for doing this is installation dependent and is described in the installation guide.

7 Check the distribution by use of the PRINT DISTRIBUTION command. The HIGH-RESOLUTION print option will print warnings if the DDI routine seems to give wrong results. Also try giving some extreme tail values using the FRACTILE and PROBABILITY options.